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LEFSCHETZ FIBRATIONS AND SYMPLECTIC HOMOLOGY

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Abstract. We show that for each $k > 3$ there are infinitely many finite type Stein manifolds diffeomorphic to Euclidean space $\mathbb{R}^{2k}$ which are pairwise distinct as symplectic manifolds.

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1. Introduction

This paper is about the symplectic topology of Stein manifolds. If we have a symplectic manifold (V, ω), then we say it carries a Stein structure if there exists a complex structure J and an exhausting (i.e. proper and bounded from below) plurisubharmonic function $\phi : V \to \mathbb{R}$ such that $\omega = -dd^c\phi$, where d^c is defined by $d^c(a)(X) := da(JX)$. The triple (V, J, ϕ) is called a Stein manifold. We say that a Stein manifold is of finite type if ϕ only has finitely many critical points, each of which is non-degenerate.

We define an equivalence relation $\sim$ on Stein manifolds by: $A \sim B$ if there exists a sequence of Stein manifolds $F_0, F_1, \cdots, F_n$ such that

(1) $F_0 = A$ and $F_n = B$
(2) F_i is either symplectomorphic or Stein deformation equivalent to F_{i+1}. (Stein deformation is defined later in Definition 2.7.)

The aim of this paper is the following theorem:

Theorem 1.1. *Let $k \geq 4$. There exists a family of finite type Stein manifolds X_i diffeomorphic to $\mathbb{R}^{2k}$ indexed by $i \in \mathbb{N}$ such that*

$$i \neq j \Rightarrow X_i \nsim X_j$$

We also have the following corollary:

Corollary 1.2. *Let M be a compact manifold of dimension 4 or higher. There exists a family of finite type Stein manifolds X_i^M diffeomorphic to T^*M indexed by $i \in \mathbb{N}$ such that*

$$i \neq j \Rightarrow X_i^M \nsim X_j^M$$

We will prove this at the end of this introduction. The results of Paul Seidel and Ivan Smith in [25] show that there exists a finite type Stein manifold diffeomorphic to $\mathbb{R}^{4k}$ but not symplectomorphic to $\mathbb{R}^{4k}, k \geq 2$. They show this by constructing an affine variety which has a Lagrangian torus which cannot be moved off itself by a Hamiltonian isotopy. In fact they show that none of these Stein manifolds can be embedded in a subcritical Stein manifold. Also, they have an argument (explained in [24]) that shows that the symplectic homology groups of these varieties are non-trivial. In this paper we strengthen this result in two ways:

(1) we give examples in all even dimensions ≥ 8 (not just in dimension $4k$ where $k \geq 2$);
(2) we also show there are countably many pairwise distinct examples in each of these dimensions.

(1) is straightforward but (2) is much harder and involves various new ideas. We show in Corollary 10.5 that these manifolds cannot be embedded in a subcritical Stein manifold.

There is no analogue of our result in dimension 4 because any finite type Stein manifold diffeomorphic to $\mathbb{R}^4$ is symplectomorphic to $\mathbb{R}^4$ (see the introduction to [25]). Having said that, Gompf in [14] constructs uncountably

many non-finite type Stein manifolds which are homeomorphic to $\mathbb{R}^4$, but are pairwise not diffeomorphic to each other. We hope to address the question of whether Theorem 1.1 holds in dimension 6, and whether we can distinguish the contact boundaries of these manifolds up to contactomorphism, in future work.

We will now construct an example of a family of Stein manifolds $(X_n)_{n\in\mathbb{N}}$ as in Theorem 1.1 in dimension 8. Let $V := \{x^7+y^2+z^2+w^2 = 0\} \subset \mathbb{C}^4$ and consider a smooth point, say $p := (0, 0, 1, i) \in V$. Let H be the blowup of $\mathbb{C}^4$ at p. Then $X := H \setminus \widetilde{V}$ is a Stein manifold where $\widetilde{V}$ is the proper transform of V. The variety X is called the Kaliman modification of $(\mathbb{C}^4, V, p)$ (see [16]). We will think of this modification in two stages:

(1) Cut out the hypersurface V in $\mathbb{C}^4$ to get $Z := \mathbb{C}^4 \setminus V$.
(2) Blow up Z at infinity to get X.

Operation (2) attaches a 2-handle along a knot which is transverse to the contact structure. All our Stein manifolds in this family will be constructed from X. If we have two Stein manifolds A and B, then it is possible to construct their *end connected sum* $A\#_e B$ (see Theorem 2.10). Roughly what we do here is join A and B with a 1-handle, and then extend the Stein structure over this handle. Finally, $X_n := \#_{i=1}^n X$ is our family of Stein manifolds.

1.1. **Sketch of the proof of the main theorem.** We will now give an outline of the proof. We will only consider the examples X_n in dimension 8 constructed above, as the higher dimensional examples are similar. For each Stein manifold Y with a trivialisation of the canonical bundle, we have an integer graded commutative $\mathbb{Z}/2\mathbb{Z}$ algebra $SH_{n+*}(Y)$ where $SH_*(Y)$ is called symplectic homology [1]. The reason why we write $SH_{n+*}(Y)$ instead of $SH_*(Y)$ is because we want the unit to be in degree 0 and not in degree n. If Y_1 and Y_2 are Stein manifolds with $H_1(Y_1) = H_1(Y_2) = 0$ and $Y_1 \sim Y_2$, then $SH_*(Y_1) = SH_*(Y_2)$ (see [24, Section 7]). The reason why we need $H_1(Y_1) = H_1(Y_2) = 0$ is because symplectic homology is only known to be invariant up to exact symplectomorphism. For each Stein manifold Y, we can define another invariant $i(Y)$ which is the number of idempotents of $SH_*(Y)$ (this invariant might be infinite). Hence all we need to do is show that for $i \neq j$, $i(X_i) \neq i(X_j)$. The next fact we need is that for any two Stein manifolds Y_1 and Y_2, $SH_*(Y_1\#_e Y_2) = SH_*(Y_1) \times SH_*(Y_2)$. This means that $SH_*(X_n) = \prod_{i=1}^n SH_*(X)$, and hence $i(X_n) = i(X)^n$. So all we need to do is show that $1 < i(X) < \infty$. If $SH_*(X) \neq 0$, $i(X) > 1$ since we have 0 and 1; but since $SH_*(X)$ can a priori be infinite dimensional in each degree, finiteness of $i(X)$ is much harder. Most of the work in this paper involves proving $i(X) < \infty$.

[1] With our convention, the pair-of-pants product makes SH_* (and not SH^*) a unital ring.

For any Stein manifold Y, $SH_*(Y)$ is $\mathbb{Z}$ graded by the Conley-Zehnder index taken with negative sign. The group $SH_*(Y)$ has a ring structure making $SH_{n+*}(Y)$ into a $\mathbb{Z}/2\mathbb{Z}$ graded algebra. This ring is also a $H_1(Y)$ graded algebra. This means that as a vector space, it is of the form $\bigoplus_{i\in H_1(Y)} R_i$ and if $x \in R_a$ and $y \in R_b$ then their product xy is in R_{a+b}. Hence idempotents in $SH_{n+*}(Y)$ are contained in $SH_n(Y)$ and are a linear combination of elements with grading in the torsion part of $H_1(Y)$ (see Lemma 7.6). The problem is that $H_1(X) = 0$. In order to find out which elements of $SH_{n+*}(X)$ are idempotents, we will show that $SH_*(X)$ is isomorphic as a ring to $SH_*(Z)$ where $Z = \mathbb{C}^4 \setminus V$ was defined above. Because Z is so much simpler than X and $H_1(Z) \neq 0$, it is possible by a direct calculation to show that $SH_{n+*}(Z)$ has finitely many idempotents.

Proving that $SH_*(X) \cong SH_*(Z)$ relies on the following theorem. This theorem is the heart of the proof. We let $E' \to \mathbb{C}$, $E'' \to \mathbb{C}$ be Lefschetz fibrations, and F' (resp. F'') be smooth fibres of E' (resp. E''). Let F' and F'' be Stein domains with F'' a holomorphic and symplectic submanifold of F'.

Theorem 1.3. *Suppose E' and E'' satisfy the following properties:*

1. *E'' is a subfibration of E'.*
2. *The support of all the monodromy maps of E' are contained in the interior of E''.*
3. *Any holomorphic curve in F' with boundary inside F'' must be contained in F''.*

Then $SH_*(E') \cong SH_*(E'')$.

Remark 1: There exist Lefschetz fibrations E', E'' with the above properties such that as convex symplectic manifolds, E' (resp. E'') is convex deformation equivalent to X (resp. Z). This is because we can choose an algebraic Lefschetz fibration on Z where the closures of all the fibres pass through p. Then blowing up Z at infinity (operation (2) of the Kaliman modification) is the same as blowing up each fibre at infinity and keeping the same monodromy. Hence $SH_*(X) \cong SH_*(Z)$.

Remark 2: Given varieties X and Z in dimension 4 such that X is obtained from Z by blowing up at infinity, there are Lefschetz fibrations E', E'' satisfying properties (1) and (2) such that as convex symplectic manifolds, E' (resp. E'') is convex deformation equivalent to X (resp. Z). These do not satisfy property (3) because E' is obtained from E'' by filling in a boundary component of the fibres with a disc.

We will prove Theorem 1.3 in two stages. In stage (i), we construct a $\mathbb{Z}/2\mathbb{Z}$ graded algebra $SH_*^{\text{lef}}(E)$ related to a Lefschetz fibration E and show it is equal to symplectic homology. This is covered in sections 4 and 5.1. In stage (ii), we prove that $SH_*^{\text{lef}}(E') \cong SH_*^{\text{lef}}(E'')$. This is covered in section 6. In a little more detail:

(i) Let F be a smooth fibre of E and $\mathbb{D}$ a disc in $\mathbb{C}$. In section 5, we show (roughly) that the chain complex C for $SH_*(E)$ is generated by:

(1) critical points of some Morse function on E;
(2) two copies of fixed points of iterates of the monodromy map around a large circle;
(3) pairs (Γ, γ) where Γ is a Reeb orbit on the boundary of F and γ is either a Reeb orbit of $\partial\mathbb{D}$ or a fixed point in the interior of $\mathbb{D}$.

This is done in almost exactly the same way as the proof of the Künneth formula for symplectic homology [19]. The differential as usual involves counting cylinders connecting the orbits and satisfying the perturbed Cauchy -Riemann equations. The orbits in (1) and (2) actually form a subcomplex C^{lef}, and we define Lefschetz symplectic homology $SH_*^{\text{lef}}(E)$ to be the homology of this subcomplex. Let $\Phi : SH_k^{\text{lef}}(E) \to SH_k(E)$ be the map corresponding to the inclusion $C^{\text{lef}} \hookrightarrow C$. We need to show that Φ is an isomorphism. In order to do this we will show that the elements of C in (3) have very high index. The orbits γ in (3) actually correspond to orbits of some Hamiltonian on the complex plane $\mathbb{C}$. We can ensure that this Hamiltonian has orbits of index as large as we like. In particular, we can assume that the index of γ is so large that the index of the pair (γ, Γ) is greater than any number we like (because the index of (γ, Γ) is the sum $\text{index}(\gamma) + \text{index}(\Gamma)$). This means for any $k \in \mathbb{Z}$, the map $SH_k^{\text{lef}}(E) \to SH_k(E)$ is an isomorphism if we ensure the indices of the orbits (γ, Γ) are all greater than $k+1$. Hence $SH_*(E) \cong SH_*^{\text{lef}}(E)$.

(ii) Let C' (resp. C'') be the chain complex for the group $SH_*^{\text{lef}}(E')$ (resp. $SH_*^{\text{lef}}(E'')$). The fibration $E' \setminus E''$ is a trivial fibration $\mathbb{D} \times W$. We have a short exact sequence $0 \to B \to C' \to C'' \to 0$ where B is generated by orbits of the form (γ, Γ) in $\mathbb{D} \times W$. The orbit Γ is a critical point of some Morse function on W and γ is either a Reeb orbit of $\partial\mathbb{D}$ or a fixed point in the interior of $\mathbb{D}$. In this case the homology of the chain complex B is actually a product $SH_*(\mathbb{D}) \otimes X$ where X is the relative cohomology group $H^{n-*}(F', F'')$. Because $SH_*(\mathbb{D}) = 0$, we have that $H_*(B) = 0$ which gives us our isomorpism between $SH_*^{\text{lef}}(E')$ and $SH_*^{\text{lef}}(E'')$. Property (3) in Theorem 1.3 is needed here to ensure that the above exact sequence exists. If we didn't have this property, then there would be some spectral sequence from $SH_*^{\text{lef}}(E'')$ (with an extra grading coming from the $H_1(E'')$ classes of these orbits) to $SH_*^{\text{lef}}(E')$.

Lefschetz symplectic homology was partially inspired by Paul Seidel's Hochshild homology conjectures [21], which also relate symplectic homology to Lefschetz fibrations. His conjectures would in particular prove Theorem 1.3.

Proof. of Corollary 1.2. There is a standard Stein structure on T^*M such that $SH_0(M)$ is a non-trivial finite dimensional $\mathbb{Z}/2\mathbb{Z}$ vector space. By Lemma 7.6 this means that $i(T^*M) < \infty$. Also $0 \in SH_*(T^*M)$ is an idempotent which means that $0 < i(T^*M)$. We let X_i be defined as in the proof of the main theorem 1.1. We define

$$X_i^M := T^*M \#_e X_i$$

Then $i(X_i^M) = i(T^*M)i(X_i)$. These numbers are all different as $0 < i(T^*M) < \infty$ and $i(X_i) \neq i(X_j)$ for $i \neq j$. □

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1.2. **Notation.** Throughout this paper we use the following notation:

(1) $M, M', M'', \dots$ are manifolds (with or without boundary).
(2) ∂M is the boundary of M.
(3) $(E, \pi), (E', \pi'), (E'', \pi'')$ are exact Lefschetz fibrations (See Definition 2.12).
(4) If we have some data X associated to M (resp. E), then $X, X', X'', \dots$ are data associated to $M, M', M'', \dots$ (resp. $E, E', E'', \dots$). For instance $\partial M''$ is the boundary of M''.
(5) ω is a symplectic form on M or E.
(6) θ is a 1-form such that $d\theta = \omega$.
(7) (M, θ) is an exact symplectic manifold.
(8) J is an almost complex structure compatible with ω.
(9) If (M, θ) is a compact convex symplectic manifold (See Definition 2.1), then $(\widehat{M}, \theta)$ is the completion of (M, θ) (Lemma 2.4). Similarly by Definition 2.16, (E, π) can be completed to $(\widehat{E}, \pi)$ (we leave π and θ as they are by abuse of notation).
(10) (M, θ_t) is a convex symplectic or Stein deformation.
(11) F will denote a smooth fibre of (E, π).
(12) If we have some subset A of a topological space, then we will let $\mathrm{nhd}(A)$ be some open neighbourhood of A.

2. Background

2.1. **Stein manifolds.** We will define Stein manifolds as in [25]. We let M be a manifold and θ a 1-form where $\omega := d\theta$ is a symplectic form.

Definition 2.1. *(M, θ) is called a* **compact convex symplectic manifold** *if M is a compact manifold with boundary and the ω-dual of θ is transverse to ∂M and pointing outwards. A* **compact convex symplectic deformation**

is a family of compact convex symplectic manifolds (M, θ_t) parameterized by $t \in [0,1]$. We will let λ be the vector field which is ω-dual to θ.

Usually, a compact convex symplectic manifold is called a convex symplectic domain. We have a natural contact form $\theta|_{\partial M}$ on ∂M, and hence we call this the contact boundary of M.

Definition 2.2. *Let M be a manifold without boundary. We say that (M, θ) is a* **convex symplectic manifold** *if there exist constants $c_1 < c_2 < \cdots$ tending to infinity and an exhausting function $\phi : M \to \mathbb{R}$ such that $(\{\phi \leq c_i\}, \theta)$ is a compact convex symplectic manifold for each i. Exhausting here means proper and bounded from below. If the flow of λ exists for all positive time, then (M, θ) is called* **complete**. *If there exists a constant $c > 0$ such that for all $x \geq c$, $(\{\phi \leq c\}, \theta)$ is a compact convex symplectic manifold, then we say that (M, θ) is of* **finite type**.

Definition 2.3. *Let (M, θ_t) be a smooth family of convex symplectic manifolds with exhausting functions ϕ_t. Suppose that for each $t \in [0,1]$, there are constants $c_1 < c_2 < \cdots$ tending to infinity and an $\epsilon > 0$ such that for each s in $(t - \epsilon, t + \epsilon)$ and $i \in \mathbb{N}$, $(\{\phi_s \leq c_i\}, \theta_s)$ is a compact convex symplectic manifold. Then (M, θ_t) is called a* **convex symplectic deformation**.

The constants $c_1 < c_2 < \cdots$ mentioned in this definition depend on t but not necessarily in a continuous way. The nice feature of convex symplectic manifolds is that we have some control over how they behave near infinity. That is, the level set $\phi^{-1}(c_k)$ is a contact manifold for all k. Also note that if we have a convex symplectic manifold of finite type then it has a cylindrical end. That is, there exists a manifold N with a contact form α such that at infinity, M is symplectomorphic to $(N \times [1, \infty), d(r\alpha))$ (r is a coordinate on $[1, \infty)$).

Lemma 2.4. *A compact convex symplectic manifold M can be completed to a finite type complete convex symplectic manifold $(\widehat{M}, \theta)$.*

This is explained for instance in [29, section 1.1]. The proof basically involves gluing a cylindrical end onto ∂M. Let (M, θ) be a complete convex symplectic manifold. Let (M', θ') be a compact convex symplectic manifold which is a codimension 0 exact submanifold of (M, θ) (i.e. $\theta|_{M'} = \theta' + dR$ for some smooth function R on M').

Lemma 2.5. *We can extend the embedding $M' \hookrightarrow M$ to an embedding $\widehat{M'} \hookrightarrow M$.*

Proof. There exists a function $R : M' \to \mathbb{R}$ such that $\theta' = \theta + dR$. We can extend R over the whole of M such that $R = 0$ outside some compact subset K containing M'. Let λ' be the ω-dual of θ'. Let $F_t : M \to M$ be the flow of λ'. This exists for all time because M is complete and $\lambda' = \lambda$ outside K. We have an embedding $\Phi : (\partial M') \times [1, \infty) \to M$ defined by $\Phi(a, t) = F_{\log t}(a)$. This attaches a cylindrical end to M' inside M, hence we have an exact embedding $\widehat{M'} \to M$ extending the embedding of M'. $\square$

Definition 2.6. *A* **Stein manifold** (M, J, ϕ) *is a complex manifold* (M, J) *with an exhausting plurisubharmonic function* $\phi : M \to \mathbb{R}$ *(i.e.* ϕ *is proper and bounded from below and* $-dd^c(\phi) > 0$ *where* $d^c = J^*d$*). A Stein manifold is called* **subcritical** *if* ϕ *is a Morse function with critical points of index* $< \frac{1}{2}\dim_{\mathbb{R}} M$*. A manifold of the form* $\phi^{-1}((-\infty, c])$ *is called a* **Stein domain**.

We can perturb ϕ so that it becomes a Morse function. From now on, if we are dealing with a Stein manifold, we will always assume that ϕ is a Morse function. The index of a critical point of ϕ is always less than or equal to $\frac{1}{2}\dim_{\mathbb{R}} M$. This is because the unstable manifolds of these critical points are isotropic submanifolds of M. Note that the definition of a Stein manifold in [11, Section 2] is that it is a closed holomorphic submanifold of $\mathbb{C}^N$ for some N. This has an exhausting plurisubharmonic function $|z|^2$. An important example of a subcritical Stein manifold is $(\mathbb{C}^n, i, |z|^2)$. The Stein manifold (M, J, ϕ) is a convex symplectic manifold $(M, \theta := -d^c\phi)$. Note that $\lambda := \nabla\phi$ where ∇ is taken with respect to the metric $\omega(\cdot, J(\cdot))$. It is easy to see that $\nabla\phi$ is a Liouville vector field transverse to a regular level set of ϕ and pointing outwards. We call a Stein manifold complete or of finite type if the associated convex symplectic structure is complete or of finite type respectively.

Definition 2.7. *If* (J_t, ϕ_t) *is a smooth family of Stein structures on* M*, then it is called a* **Stein deformation** *if the function* $(t, x) \longrightarrow \phi_t(x)$ *is proper and for each* $t \in [0, 1]$*, there exists* $c_1 < c_2 < \dots$ *tending to infinity and an* $\epsilon > 0$ *such that for each* s *in* $(t - \epsilon, t + \epsilon)$ *we have that* c_k *is a regular value of* ϕ_s*. This induces a corresponding convex symplectic deformation.*

Example 2.8. *An affine algebraic subvariety* M *of* $\mathbb{C}^N$ *admits a Stein structure. This is because it has a natural embedding in* $\mathbb{C}^N$*, so we can restrict the plurisubharmonic function* $\|z\|^2$ *to this variety to make it into a Stein manifold. We can also use the following method to find a plurisubharmonic function on* M*. We first compactify* M *by finding a projective variety* X *with complex structure* i *and an effective ample divisor* D *such that* $M = X \setminus D$ *(for instance we can embed* M *in* $\mathbb{C}^N \subset \mathbb{P}^N$ *and then let* X *be the closure of* M *in* $\mathbb{P}^N$*). There exists an ample line bundle* $E \longrightarrow X$ *associated to the divisor* D*. Choose a holomorphic section* s *of* E *such that* $D = s^{-1}(0)$*. Then ampleness means that we can choose a metric* $\|.\|$ *such that its curvature form* $\omega := iF_\nabla$ *is a positive* $(1, 1)$*-form. Hence we have a Stein structure*

$$(M := X \setminus D, J := i, \phi := -\log\|s\|)$$

Note that by [25, Lemma 8]*, this is of finite type.*

For affine algebraic varieties we need a uniqueness theorem. This basically comes from the text following [24, Lemma 4.4]. The problem is that this text only deals with the case when our compactification divisor D is a normal crossing divisor. But the methods used carry over to general divisors by

using Hironaka's resolution theorem. In fact, the paper [24] uses Hironaka's resolution theorem in the same way as we will use it. We will repeat the argument here. Let M be an algebraic variety.

Lemma 2.9. *If h_1 and h_2 are Stein functions on M constructed as in the previous example, then $(1-t)h_1 + th_2$ is a Stein deformation.*

Proof. If we can prove that all the critical points of $(1-t)h_1+th_2$ stay inside a compact set for all $t \in [0,1]$, then this will prove that it is a Stein deformation. The way we do this is to look at this function in a neighbourhood of some compactification divisor of M. For $i = 1, 2$, let X_i be projective varieties compactifying M. Let E_i be ample line bundles on X_i whose associated divisor is effective and ample and has support equal to $X_i \setminus M$. Suppose h_i is equal to $-\log\|s_i\|_i$ where s_i is a section of the line bundle E_i such that $s_i^{-1}(0)$ has support equal to $X_i \setminus M$ and $\|\cdot\|_i$ is a metric on E_i.

By Hironaka's resolution theorem, we have that there exists a compactification X by a normal crossing divisor D such that it dominates X_1 and X_2 (i.e there is a surjective morphism $X \to X_i$ such that away from D and D_i it is an isomorphism). We pull back the line bundles, sections and metrics to X and Stein functions via these morphisms. We now just apply [24, Lemma 4.4], and this gives us our result. □

The following operation constructs a new Stein manifold from two old ones. This is used to construct our infinite family of Stein manifolds. We will let $(M, J, \phi), (M', J', \phi')$ be complete finite type Stein manifolds. Because these manifolds are complete and of finite type, they are the completions of compact convex symplectic manifolds N, N' respectively. In fact $N = \{\phi \leq R\}, N' = \{\phi' \leq R\}$ for some arbitrarily large R. Let p (resp. p') be a point in ∂N (resp. $\partial N'$). The following theorem is proved in greater generality in [10] and [8, Theorem 9.4].

Theorem 2.10. *There exists a connected finite type Stein manifold (M'', J'', ϕ'') such that $N'' := \{\phi'' \leq R\}$ is biholomorphic to the disjoint union of N and N' with $\phi''|_N = \phi$ on N and $\phi''|_{N'} = \phi'$ on N'. Also, the only critical point of ϕ'' outside N'' has index* 1.

In this theorem, what we are doing is joining N and N' with a 1-handle and then extending the Stein structure over this handle, and then completing this manifold so that it becomes a Stein manifold. The Stein manifold M'' is called the **end connect sum** of M and M', and we define $M\#_e M'$ as this end connected sum. If M and M' are Stein manifolds diffeomorphic to $\mathbb{C}^n$, then $M\#_e M'$ is also diffeomorphic to $\mathbb{C}^n$.

Remark: There is an example due to the author of a non-finite type Stein manifold which is not equivalent to any finite type Stein manifold. This is described by Seidel in [24, Section 7], and is constructed as an infinite end-connect-sum.

2.2. **Lefschetz fibrations.** Throughout this section we will let E be a compact manifold with corners whose boundary is the union of two faces $\partial_h E$ and $\partial_v E$ meeting in a codimension 2 corner. We will also assume that Ω is a 2-form on E and Θ a 1-form satisfying $d\Theta = \Omega$. We let S be a surface with boundary. Let $\pi : E \to S$ be a smooth map with only finitely many critical points (i.e. points where $d\pi$ is not surjective). Let $E^{\text{crit}} \subset E$ be the set of critical points of π and $S^{\text{crit}} \subset S$ the set of critical values of π. For $s \in S$, let E_s be the fibre $\pi^{-1}(s)$.

Definition 2.11. *If for every $s \in S$ we have that Ω is a symplectic form on $E_s \setminus E^{\text{crit}}$ then we say that Ω* **is compatible with** *π.*

Note that if Ω is compatible with π then there is a natural connection (defined away from the critical points) for π defined by the horizontal plane distribution which is Ω-orthogonal to each fibre. If parallel transport along some path in the base is well defined then it is an exact symplectomorphism (an exact symplectomorphism is a diffeomorphism Φ between two symplectic manifolds $(M_1, d\theta_1)$ and $(M_2, d\theta_2)$ such that $\Phi^*\theta_2 = \theta_1 + dG$ where G is a smooth function on M_1). From now on we will assume that Ω is compatible with π. We deal with Lefschetz fibrations as defined in [22]. We define F to be some smooth fibre of π.

Definition 2.12. (E, π) *is an* **exact Lefschetz fibration** *if:*

1. *$\pi : E \longrightarrow S$ is a proper map with $\partial_v E = \pi^{-1}(\partial S)$ and such that $\pi|_{\partial_v V} : \partial_v E \to \partial S$ is a smooth fibre bundle. Also there is a neighbourhood N of $\partial_h E$ such that $\pi|_N : N \to S$ is a product fibration $S \times nhd(\partial F)$ where $\Omega|_N$ and $\Theta|_N$ are pullbacks from the second factor of this product.*
2. *There is an integrable complex structure J_0 (resp. j_0) defined on some neighbourhood of E^{crit} (resp. S^{crit}) such that π is (J_0, j_0) holomorphic near E^{crit}. At any critical point, the Hessian $D^2\pi$ is nondegenerate as a complex quadratic form. We also assume that there is at most one critical point in each fibre.*
3. *Ω is a Kähler form for J_0 near E^{crit}.*

Sometimes we will need to define a Lefschetz fibration without boundary. This is defined in the same way as an exact Lefschetz fibration except that E, the fibre F and the base S are open manifolds without boundary. We replace "neighbourhood of $\partial_h E$" in the above definition with an open set whose complement is relatively compact when restricted to each fibre. We also replace "$\partial_v E$" with $\pi^{-1}(S \setminus K)$ where K is a compact set in S. Also π is obviously no longer a proper map, and we assume that the set of critical points is compact. From now on we will let (E, π) be an exact Lefschetz fibration.

Lemma 2.13. [22, Lemma 1.5] *If β is a symplectic form on S then $\omega := \Omega + N\pi^*\beta$ is a symplectic form on E for N sufficiently large.*

We really want our Lefschetz fibrations to be described as finite type convex symplectic manifolds.

Definition 2.14. *A* **compact convex Lefschetz fibration** *is an exact Lefschetz fibration (E, π) such that $(F, \Theta|_F)$ is a compact convex symplectic manifold. A* **compact convex Lefschetz deformation** *is a smooth family of compact convex Lefschetz fibrations parameterized by $[0, 1]$.*

Note that by the triviality condition at infinity, all smooth fibres of π are compact convex symplectic manifolds as long as the base S is connected. From now on we will assume that (E, π) is a compact convex Lefschetz fibration.

Theorem 2.15. *Let the base S be a compact convex symplectic manifold (S, θ_S). There exists a constant $K > 0$ such that for all $k \geq K$ we have: $\omega := \Omega + k\pi^*(\omega_S)$ is a symplectic form, and the ω-dual λ of $\Theta + k\pi^*\theta_S$ is transverse to ∂E and pointing outwards.*

(The proof is given in section 4.) Note that this theorem also implies that if we have a compact convex Lefschetz deformation, then we have a corresponding compact convex symplectic deformation because we can smooth the codimension 2 corners slightly.

If we have a compact convex Lefschetz fibration, then we wish to extend the Lefschetz fibration structure over the completion $\widehat{E}$ of E. Here is how we naturally complete (E, π): The horizontal boundary is a product $\partial F \times S$. We can add a cylindrical end $G := (\partial F \times [1, \infty)) \times S$ to this in the usual way (i.e. as in Lemma 2.4), extending Θ over this cylindrical end by the 1-form $r(\Theta|_{\partial F})$ where r is the coordinate for $[1, \infty)$. Let E_1 be the resulting manifold. We also extend the map π over E_1 by letting $\pi|_G : G \to S$ be the natural projection. This ensures that π is compatible with the natural symplectic form on E_1 defined as in Lemma 2.13. The fibres of π are finite type complete convex symplectic manifolds. We now need to "complete" the vertical boundary of E_1 so that we have a fibration over the completion $\widehat{S}$ of S. Let $V := \partial_v E_1 := \pi^{-1}(\partial S)$. We then attach $A := V \times [0, \infty)$ to E_1 by identifying $V \subset E_1$ with $V \times \{0\} \subset A$ to create a new manifold $\widehat{E}$. We can extend our map π over A by letting $\pi|_A(v, r) = \pi|_V(v)$ where $v \in V \subset E_1$ and $r \in [0, \infty)$. Let $\pi_1 : A \to V$ be the natural projection onto V. Let $\Theta_A := \pi_1^*\Theta|_V$. We wish to join Θ with Θ_A inside $\widehat{E}$. Let $\zeta : [0, \infty) \to [0, 1]$ be a smooth function such that $\zeta(x) = 0$ for $x < 0.5$ and $\zeta(x) = 1$ for $x > 1$. The 1-form Θ smoothly extends to a 1-form Q defined on $V \times [0, \epsilon] \subset A$ such that dQ restricted to the fibres of π is a symplectic form and $Q = \Theta$ in the region $G' = (\partial F \times [1, \infty)) \times \widehat{S}$ (i.e. the region where the Lefschetz fibration is a product near infinity). Let $\Theta'_A(v, r) := (1 - \zeta(Kr/\epsilon))Q + \zeta(Kr/\epsilon)\Theta_A$ where K is a large constant. We can now smoothly extend Θ over A by the 1-form Θ'_A. Note that $d\Theta'_A$ is a symplectic form on all the fibres for K large enough because $\Theta|_V = \Theta_A|_{V \times \{0\}}$, and hence dQ restricted to the fibres is as close as we like to $d\Theta_A$ restricted to the fibres. This means that

$(1-t)dQ + td\Theta_A$ is arbitrarily close to a symplectic form when restricted to the fibres and hence it is a symplectic form when restricted to the fibres for $t \in [0,1]$ and this implies that $d\Theta'_A$ is a symplectic form on all the fibres.

Definition 2.16. *$(\widehat{E}, \pi)$ is called the* **completion** *of (E, π).*

Note that the base of our completed fibration is $(\widehat{S}, \theta_S)$.

Definition 2.17. *Any fibration which is the completion of a compact convex Lefschetz fibration is called a* **complete convex Lefschetz fibration**.

Note that if we add a large multiple of $\pi^*\theta_S$ to Θ then $(\widehat{E}, \theta)$ is a complete finite type convex symplectic manifold. Lefschetz fibrations have well defined parallel transport maps due to the fact that the fibration is trivial near the horizontal boundary of E. Now we need to deal with almost complex structures on $\widehat{E}$, as this will be useful when we later define SH^{lef}_*. Let J (resp. j) be an almost complex structure on $\widehat{E}$ (resp. $\widehat{S}$).

Definition 2.18. *We say that (J, j) are* **compatible** *with $(\widehat{E}, \pi)$ if:*

1. *π is (J, j)-holomorphic, and that $J = J_0$ near E^{crit} and $j = j_0$ near S^{crit}.*
2. *j is convex at infinity with respect to the convex symplectic structure of $\widehat{S}$ (i.e. $\theta_S \circ j = dr$ for large r where θ_S is the contact form at infinity on $\widehat{S}$ and r is the radial coordinate of the cylindrical end of $\widehat{S}$).*
3. *J is a product (J_F, j) on the region $C \times S$ where C is the cylindrical end $\partial F \times [1, \infty)$ of $\widehat{F}$, and J_F is convex at infinity for F. ($\theta|_{\widehat{F}} \circ J_F = dr$ for large r where r is the radial coordinate of the cylindrical end).*
4. *$\omega(\cdot, J\cdot)$ is symmetric and positive definite.*

If $(\widehat{E}, \pi)$ is a complete convex Lefschetz fibration then the space of such almost complex structures is nonempty and contractible (see [22, section 2.2]). We wish to have a slightly larger class of almost complex structures.

Definition 2.19. *We define $\mathcal{J}^h(\widehat{E})$ to be the space of almost complex structures on $\widehat{E}$ such that for each J in this space, there exists a (J_1, j_1) compatible with $(\widehat{E}, \pi)$ and a compact set $K \subset \widehat{E}$ with $J = J_1$ outside K and with $\omega(\cdot, J\cdot)$ symmetric and positive definite everywhere.*

The set of such complex structures is still contractible.

2.3. **Symplectic homology.** In this section we will discuss symplectic homology as defined by Viterbo in [29] for finite type Stein manifolds. For simplicity we will assume that our homology theory has coefficients in $\mathbb{Z}/2$.

Let (M, θ) be a compact convex symplectic manifold. Our manifold $\widehat{M}$ has a cylindrical end symplectomorphic to $(N \times [1, \infty), d(r\alpha))$ where r is a coordinate on $[1, \infty)$ and α is a contact form on N. We choose a smooth

function $H : \mathbb{S}^1 \times \widehat{M} \longrightarrow \mathbb{R}$ such that each H_t is of the form $H_t = ar + b$ at infinity where a and b are some constants independent of t. We call such a Hamiltonian **admissible**. We also choose an $\mathbb{S}^1$ family of almost complex structures J_t compatible with the symplectic form. We assume that J_t is convex with respect to this cylindrical end outside some large compact set (i.e. $\theta \circ J_t = dr$). We call the constant a the **slope at infinity**. We also say that J_t is **admissible**.

We define an $\mathbb{S}^1$ family of vector fields X_{H_t} by $\omega(X_{H_t}, \cdot) = dH_t(\cdot)$. The flow $\mathrm{Flow}^t_{X_{H_t}}$ is called the **Hamiltonian flow**. A **1-periodic orbit** of H_t is a path of the form $l : [0,1] \to \widehat{M}$, $l(t) := \mathrm{Flow}^t_{X_{H_t}}(p)$ where $p \in \widehat{M}$ and $l(0) = l(1)$. We choose the cylindrical end and the slope of our Hamiltonians so that the union of the 1-periodic orbits form a compact set. Let $F := \mathrm{Flow}^1_{X_{H_t}}$, then we have a correspondence between 1-periodic orbits x and fixed points p of F. In particular we say that x is non-degenerate if $DF|_p : T_p\widehat{M} \to T_p\widehat{M}$ has no eigenvalue equal to 1. We can also assume that the 1-periodic orbits of our Hamiltonian flow X_{H_t} are non-degenerate. We call a Hamiltonian H satisfying these conditions a **non-degenerate admissible Hamiltonian** or an admissible Hamiltonian with non-degenerate orbits.

From now on we will assume that $c_1(M) = c_1(\widehat{M}) = 0$. If we are given a trivialisation of the canonical bundle $\mathcal{K} \cong \mathcal{O}$, then for each orbit x, we can define an index of x called the **Robbin-Salamon** index (This is equal to the Conley-Zehnder index taken with negative sign). The choice of these indices depend on the choice of trivialisation of $\mathcal{K}$ up to homotopy but the indices are canonical if $H_1(M) = 0$. We denote this index by $\mathrm{ind}(x)$. Let

$$CF_k(M, H, J) := \bigoplus_{\mathrm{Flow}^1_{X_{H_t}}(x)=x, ind(x)=k} \mathbb{Z}/2\langle x \rangle$$

For a 1-periodic orbit γ we define the **action** $A_H(\gamma)$:

$$A_H(\gamma) := -\int_0^1 H(t, \gamma(t))dt - \int_\gamma \theta$$

This is the convention of [29] and [19]. This differs in sign from Seidel's convention in [24]. We will now describe the differential

$$\partial : CF_k(M, H, J) \to CF_{k-1}(M, H, J)$$

We consider curves $u : \mathbb{R} \times \mathbb{S}^1 \longrightarrow \widehat{M}$ satisfying the perturbed Cauchy-Riemann equations:

$$\partial_s u + J_t(u(s,t))\partial_t u = \nabla^{g_t} H$$

where ∇^{g_t} is the gradient associated to the $\mathbb{S}^1$ family of metrics $g_t := \omega(\cdot, J_t(\cdot))$. For two periodic orbits x_-, x_+ let $\bar{U}(x_-, x_+)$ denote the set of all curves u satisfying the Cauchy-Riemann equations such that $u(s, \cdot)$ converges to $x_\pm$ as $s \to \pm\infty$. This has a natural $\mathbb{R}$ action given by replacing the coordinate s with $s + v$ for $v \in \mathbb{R}$. Let $U(x_-, x_+)$ be equal

to $\bar{U}(x_-, x_+)/\mathbb{R}$. For a C^∞ generic admissible Hamiltonian and complex structure we have that $U(x_-, x_+)$ is an $\text{ind}(x_-) - \text{ind}(x_+) - 1$ dimensional manifold (see [12]). There is a maximum principle which ensures that all elements of $U(x_-, x_+)$ stay inside a compact set K (see [18, Lemma 1.5]). Hence we can use a compactness theorem (see for instance [4]) which ensures that if $\text{ind}(x_-) - 1 = \text{ind}(x_+)$, then $U(x_-, x_+)$ is compact and hence a finite set. Let $\#U(x_-, x_+)$ denote the number of elements of $U(x_-, x_+)$ mod 2. Then we have a differential:

$$\partial : CF_k(M, H, J) \longrightarrow CF_{k-1}(M, H, J)$$

$$\partial\langle x_-\rangle := \sum_{\text{ind}(x_+)=\text{ind}(x_-)-1} \#U(x_-, x_+)\langle x_+\rangle$$

By analysing the structure of 1-dimensional moduli spaces, one shows $\partial^2 = 0$ and defines $SH_*(M, H, J)$ as the homology of the above chain complex. As a $\mathbb{Z}/2\mathbb{Z}$ module $CF_k(M, H, J)$ is independent of J, but its boundary operator does depend on J. The homology group $SH_*(M, H, J)$ depends on M, H but is independent of J up to canonical isomorphism. Note that for each $f \in \mathbb{R}$ we have a subcomplex generated by orbits of action $\leq f$. The reason why this is a subcomplex is because if $\bar{U}(x_-, x_+)$ is a non-empty set, then $A_H(x_-) \geq A_H(x_+)$. This means that the differential decreases action and hence if x_- is an orbit of action $\leq f$, then $\partial(x_-)$ is a linear combination of orbits of action $\leq f$. The homology of such a complex is denoted by: $SH_*^{\leq f}(M, H, J)$.

If we have two non-degenerate admissible Hamiltonians $H_1 \leq H_2$ and two admissible almost complex structures J_1, J_2, then there is a natural map:

$$SH_*(M, H_1, J_1) \longrightarrow SH_*(M, H_2, J_2)$$

This map is called a **continuation map**. This map is defined from a map C on the chain level as follows:

$$C : CF_k(M, H_1, J_1) \longrightarrow CF_k(M, H_2, J_2)$$

$$\partial\langle x_-\rangle := \sum_{\text{ind}(x_+)=\text{ind}(x_-)} \#P(x_-, x_+)\langle x_+\rangle$$

where $P(x_-, x_+)$ is the set of solutions of the following equations: Let K_s be a smooth increasing family of admissible Hamiltonians joining H_1 and H_2 and J_s a smooth family of admissible almost complex structures joining J_1 and J_2. Then for a C^∞ generic family (K_s, J_s), the set $P(x_-, x_+)$ is the set of solutions to the parameterized Floer equations

$$\partial_s u + J_{s,t}(u(s,t))\partial_t u = \nabla^{g_t} K_{s,t}$$

such that $u(s, \cdot)$ converges to $x_\pm$ as $s \to \pm\infty$. If we have another such increasing family admissible Hamiltonians joining H_1 and H_2 and another smooth family of admissible almost complex structures joining J_1 and J_2, then the continuation map induced by this second family is the same as the

map induced by (K_s, J_s). The composition of two continuation maps is a continuation map.

If we take the direct limit of all these maps with respect to admissible Hamiltonians ordered by $\leq$, then we get our symplectic homology groups $SH_*(M)$. Supposing we have a family of Hamiltonians $(H_\lambda)_{\lambda\in\Lambda}$ ordered by $\leq$. We say that a family of Hamiltonians $(H_i)_{i\in I\subset\Lambda}$ is *cofinal* if for every $\lambda \in \Lambda$, there exists an $i \in I$ and a constant a such that $H_\lambda \leq H_i + a$. The fact that continuation maps are natural ensures that we can also define $SH_*(M)$ as the direct limit of all these maps with respect to any cofinal family of Hamiltonians. If we have a degenerate admissible Hamiltonian H, then we can still define symplectic Homology $SH_*(H, J)$ as a direct limit $SH_*(H_k, J_k)$ where H_k and J_k are C^∞ generic making $SH_*(H_k, J_k)$ well defined, and such that H_k and J_k tend to H and J. The limit is taken with respect to continuation maps. This is independent of choices of H_k and J_k due to the naturality of continuation maps (see [29, Remark 1.2]).

The symplectic homology groups also have a ring structure. The product is called the pair of pants product and it makes $SH_{n+*}(M)$ into a $\mathbb{Z}/2\mathbb{Z}$ graded algebra where n is the complex dimension of M. The product is defined in the following way: Take an admissible Hamiltonian H_1 and another admissible Hamiltonian H_2 such that $H_1 = H_2$ outside some large compact set, and such that $H_1(1, \cdot) = H_2(0, \cdot)$ with all time derivatives. Then $H_1\#H_2$ is also admissible where $H_1\#H_2$ is defined as:

$$H_1\#H_2(t, x) := \begin{cases} 2H_1(2t, x) & (t \in [0, \frac{1}{2}]) \\ 2H_2(2t - 1, x) & (t \in [\frac{1}{2}, 1]) \end{cases}$$

We define a chain map (maybe after perturbing the Hamiltonians $H_1, H_2, H_1\#H_2$ slightly):

$$CF_k(H_1, J) \otimes CF_j(H_2, J) \to CF_{k+j-n}(H_1\#H_2, J),$$

$$x_1 \otimes x_2 \to \sum_{\mathrm{ind}(x_3)=k+j-n} \#\mathcal{M}(H, J, x_1, x_2, x_3)\langle x_3\rangle.$$

The set $\mathcal{M}(H, J, x_1, x_2, x_3)$ is the set of maps $u : \mathbb{P}^1 \setminus \{0, 1, \infty\} \to M$ which satisfy some Floer type equations and such that each puncture converges to an orbit. The details of this are explained in [1], [2] and [24, Section 8]. This commutes with continuation maps and hence we can take the direct limit of these maps with respect to the ordering $\leq$ giving us a map:

$$SH_{n+i}(M) \otimes SH_{n+j}(M) \to SH_{n+i+j}(M).$$

This also has a unit in degree n and is given by counting holomorphic planes ([24, Section 8]).

Suppose that (M', θ') is a compact convex symplectic manifold which is an exact submanifold of M, then there exists a natural map

$$i : SH_*(M) \longrightarrow SH_*(M')$$

called the transfer map. The composition of two of these transfer maps is another transfer map. These maps are introduced in [29, Section 2] and studied in [7, Section 3.3]. Let N be a general convex symplectic manifold. Let $(W_j)_{j\in J}$ be the set of codimension 0 compact convex exact symplectic submanifolds. This is a directed system, where the morphisms are just inclusion maps. If $W_{j_1} \subset W_{j_2}$, then we have a transfer map $SH_*(W_{j_2}) \to SH_*(W_{j_1})$. Because the transfer map points in the opposite direction (i.e. from W_{j_2} to W_{j_1} instead of W_{j_1} to W_{j_2}), we have an inverse system $(SH_*(W_j))_{j\in J}$ where the morphisms now are transfer maps. Hence we can define the ring $SH_*(N)$ as the inverse limit of this inverse system. Because the definition only involves codimension 0 exact symplectic manifolds, we have that it is invariant under exact symplectomorphism. We have that $SH_*(N)$ is invariant under convex symplectic deformations (see [17, Theorem 2.12]). We also have that for a compact convex symplectic manifold M, $SH_*(M) = SH_*(\widehat{M})$ (see [17, Theorem 2.12]).

Symplectic homology $SH_*(M)$ also has a natural $H_1(M)$ grading. Each orbit x of some Hamiltonian H is contained in some $H_1(M)$ class, and the definition of the differential ensures that ∂x is a linear combination of orbits in the same $H_1(M)$ class. Also if we have two orbits x_1 and x_2 of some Hamiltonian H_1 and H_2 respectively, then they are in $H_1(M)$ classes a and b. If we have a pair of pants, where two of the boundaries are in classes a and b, then the third boundary is in the class $a + b$ (if we orient the boundaries correctly). This ensures that the pants product of x_1 and x_2 is a linear combination of orbits of $H_1 \# H_2$ in the homology class $a+b$. Hence the pair of pants product for $SH_*(M)$ is additive in $H_1(M)$.

Theorem 2.20. *Let M, M' be finite type Stein manifolds of real dimension greater than 2, then*
$SH_*(M\#_e M') \cong SH_*(M) \times SH_*(M')$ as rings. Also the transfer map $SH_*(M\#_e M') \to SH_*(M)$ is just the natural projection

$$SH_*(M) \times SH_*(M') \twoheadrightarrow SH_*(M).$$

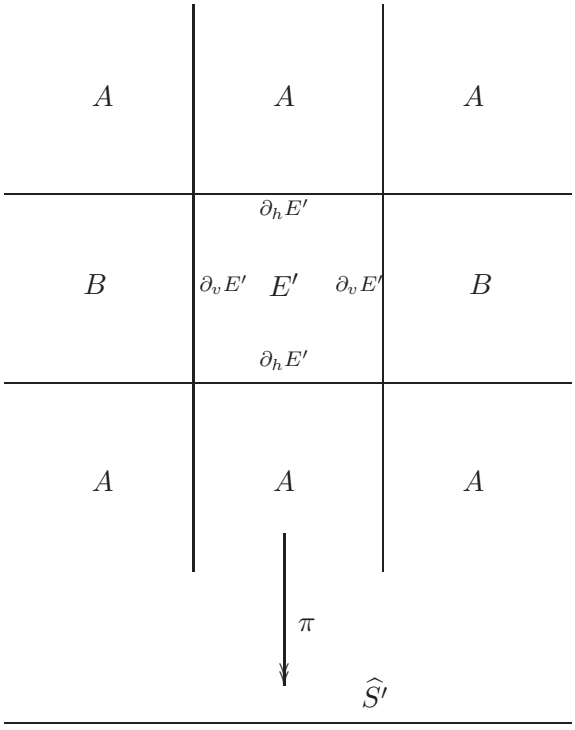
Cieliebak in [7] showed that the above theorem is true if we view SH_* as a vector space. We will prove that we have a ring isomorphism in section 10.3.

2.4. **Symplectic homology and Lefschetz fibrations.** We need three theorems which relate symplectic homology to Lefschetz fibrations. These are the key ingredients in proving that our exotic Stein manifolds are pairwise distinct. The proofs of these theorems will be deferred to sections 5 and 6. Theorem 2.22 is very close to Oancea's Künneth formula [19] but theorems 2.24 and 2.25 are new and the main part of the story. Throughout this section we will let $\pi' : E' \to S'$ be a compact convex Lefschetz fibration with fibre F'. From now on we will assume that $c_1(E') = 0$ and to make $SH_*(E')$ graded we will choose a trivialisation of the canonical bundle of

E'. Note that when we talk about symplectic homology of a compact convex Lefschetz fibration, we mean the symplectic homology of its completion with respect to the convex symplectic structure. The fibration $\widehat{E'}$ can be partitioned into three sets as follows:

(1) $E' \subset \widehat{E'}$
(2) $A := F'_e \times \widehat{S'}$, where $F'_e := \partial F' \times \mathbb{R}_{\geq 1}$ is the cylindrical end of $\widehat{F'}$.
(3) $B := \widehat{E'} \setminus (A \cup E')$

The set B is of the form $(A_1 \times \mathbb{R}_{\geq 1}) \bigsqcup (A_2 \times \mathbb{R}_{\geq 1}) \cdots \bigsqcup (A_n \times \mathbb{R}_{\geq 1})$, where A_i is a mapping torus of the monodromy symplectomorphism around one of the boundary components of S'. Here is a picture of the regions E', A and B.



Let $\pi_1 : A \twoheadrightarrow F'_e$ be the natural projection onto F'_e.

Definition 2.21. *Let $H_{S'}$ be an admissible Hamiltonian for the base $\widehat{S'}$. Let $H_{F'}$ be an admissible Hamiltonian for the fibre $\widehat{F'}$. We assume that $H_{F'} = 0$ on $F' \subset \widehat{F'}$. The map $H : \widehat{E'} \to \mathbb{R}$ is called a* **Lefschetz admissible Hamiltonian** *if $H|_A = \pi^* H_{S'} + \pi_1^* H_{F'}$ and $H|_B = \pi^* H_{S'}$ outside some large compact set. We say that H has* **slope** (a, b) *if $H_{S'}$ has slope a at infinity and H_F has slope b at infinity.*

Let H be a Lefschetz admissible Hamiltonian and let J be an admissible almost complex structure for E'. We will call the pair (H, J) a Lefschetz admissible pair. For generic (H, J) we can define $SH_*(E', H, J)$ (see section 5 for more details). If (H_1, J_1) is another generic Lefschetz admissible pair such that $H \leq H_1$, then there is a continuation map $SH_*(H, J) \to SH_*(H_1, J_1)$ induced by an increasing homotopy from H to H_1 through Lefschetz admissible Hamiltonians. Hence, we have a direct limit $SH^l_*(E') := \varinjlim_{(H,J)} SH_*(H, J)$ with respect to the ordering $\leq$ on Hamiltonians H. This has the natural structure of a ring with respect to the pair of pants product.

Theorem 2.22. *There is a ring isomorphism $SH_*(E') \cong SH^l_*(E')$.*

This will be proved in section 5. Let ϵ be smaller than the length of the shortest Reeb orbit of $\partial F'$. A Hamiltonian H is called a **half admissible Hamiltonian** if it is Lefschetz admissible and has slope (a, ϵ). We let J be an admissible almost complex structure for E'.

Definition 2.23. *We define*

$$SH^{\text{lef}}_*(E') := \varinjlim_{(H,J)} SH_*(H, J)$$

as the direct limit with respect to the ordering $\leq$ on half admissible Hamiltonians H. This has the structure of a ring as usual.

The difference between $SH_*(E')$ and $SH^l_*(E')$ is that $SH_*(E')$ is defined using Hamiltonians which are linear with respect to some fixed cylindrical end. The ring $SH^l_*(E')$ is defined using Hamiltonians which are linear in the horizontal and vertical directions with respect to some Lefschetz fibration. The difference between $SH^{\text{lef}}_*(E')$ and the other homology theories is that the slopes of a cofinal family of half admissible Hamiltonians do not have to tend to infinity pointwise in the vertical direction. This has to be true for $SH_*(E')$ and $SH^l_*(E')$ where the Hamiltonians have to get steeper and steeper at infinity in all directions. Because a half admissible Hamiltonian is Lefschetz admissible, we have a natural ring homomorphism:

$$\Phi : SH^{\text{lef}}_*(E') \to SH^l_*(E').$$

This comes from continuation maps $SH_*(H, J) \to SH_*(K, J)$ where H is a half admissible Hamiltonian and K is a Lefschetz admissible Hamiltonian. This is because $H \leq K$ when K is large enough.

Theorem 2.24. *If $S' = \mathbb{D}$, the unit disk, then Φ is an isomorphism of rings. Hence by Theorem 2.22,*

$$SH_*(E') \cong SH^{\text{lef}}_*(E')$$

as rings.

This will be proved in section 5.1. Let F' (resp. F'') be a smooth fibre of E' (resp. E''). Let F' and F'' be Stein domains with F'' a holomorphic and symplectic submanifold of F'.

Theorem 2.25. *Suppose E' and E'' satisfy the following properties:*

1. *E'' is a subfibration of E' over the same base.*
2. *The support of all the monodromy maps of E' are contained in the interior of E''.*
3. *Any holomorphic curve in F' with boundary inside F'' must be contained in F''.*

Then $SH_*^{\mathrm{lef}}(E') \cong SH_*^{\mathrm{lef}}(E'')$ as rings.

This theorem will be proved in section 6. Combining this theorem with Theorem 2.24 proves the key theorem 1.3 in the introduction of this paper.

2.5. **The Kaliman modification.** In order to produce examples of exotic symplectic manifolds, we first need to construct exotic algebraic varieties. One tool used for constructing these manifolds is called the Kaliman modification. Our treatment follows section 4 of [30].

Consider a triple (M, D, C) where $C \subseteq D \subseteq M$ are complex varieties. Let M and C be smooth, D be an irreducible hypersurface in M, and C be a closed subvariety contained in the smooth part of D such that $\dim(C) < \dim(D)$.

Definition 2.26. *(see* [16]*) The* **Kaliman modification** *M' of (M, D, C) is defined by $M' := \mathrm{Kalmod}\,(M, D, C) = \tilde{M} \setminus \tilde{D}$ where $\tilde{M}$ is the blowup of M along C and $\tilde{D}$ is the proper transform of D in $\tilde{M}$.*

The Kaliman modification of an affine variety is again an affine variety (see [16]).

Lemma 2.27. [16, Theorem 3.5] *Suppose that (i) D is a topological manifold, and (ii) D and C are acyclic. Then M' is contractible iff M is.*

Example 2.28. *(***tom Dieck-Petrie surfaces** *see* [27, 26]*) For $k > l \geq 2$ with (k, l) coprime, the triple $A_{k,l} := \left(\mathbb{C}^2, \{x^k - y^l = 0\}, \{(1,1)\}\right)$ satisfies the conditions of Lemma 2.27. Hence $X_{k,l} = \mathrm{Kalmod}\,(A_{k,l})$ is contractible. Note: $X_{k,l}$ is isomorphic to*

$$\left\{ \frac{(xz+1)^k - (yz+1)^l - z}{z} = 0 \right\}.$$

Here x, y, z are the standard coordinates of $\mathbb{C}^3$. Also the numerator of this fraction is divisible by z, hence the above fraction is a polynomial.

Here is another construction:

Example 2.29. *(***Kaliman** [16]*) If we have a contractible affine variety M of complex dimension n, then we can construct a contractible affine variety*

$$M_k := \mathrm{Kalmod}\,(M \times \mathbb{C}, M \times \{p_1, \ldots, p_k\}, \{(a_1, p_1), \ldots, (a_k, p_k)\})$$

where p_i are distinct points in $\mathbb{C}$ and a_i are points in M. This variety is contractible by a repeated application of Lemma 2.27, because it is a repeated Kaliman modification with D isomorphic to M and C a point. There are obvious variants: replace $\mathbb{C}$ and $\{p_1, \dots, p_k\}$ with some contractible variety and a disjoint union of contractible irreducible hypersurfaces, etc.

At the moment we are only discussing contractibility of varieties. We need to produce varieties diffeomorphic to some $\mathbb{C}^n$. We will use the h-cobordism theorem to achieve this stronger condition.

Corollary 2.30. *(See* [6, Page 174]*,* [20] *and* [30, Proposition 3.2]*) Let M be a contractible Stein manifold of finite type. If $n := \dim_{\mathbb{C}} M \geq 3$ then M is diffeomorphic to $\mathbb{C}^n$.*

Proof. Let (J, ϕ) be the Stein structure associated with M. We can also assume that ϕ is a Morse function. For R large enough, the domain $M_R := \{\phi < R\}$ is diffeomorphic to the whole of M as M is of finite type. We want to show that the boundary of $\bar{M}_R := \{\phi \leq R\}$ is simply connected, then the result follows from the h-cobordism theorem.

The function $\psi := R - \phi$ only has critical points of index $\geq n \geq 3$ because the function ϕ only has critical points of index $\leq n$ (see [11, Corollary 2.9]). Viewing ψ as a Morse function, $\bar{M}_R$ is obtained from $\partial \bar{M}_R$ by attaching handles of index ≥ 3. This does not change π_1, hence $\partial \bar{M}_R$ is simply connected because $\bar{M}_R$ is simply connected. □

We now need a theorem which relates the Kaliman modification with symplectic homology. We do this via Lefschetz fibrations. Let X,D,M be as in Example 2.8. Let Z be an irreducible divisor in X and $q \in (Z \cap M)$ a point in the smooth part of Z. We assume there is a rational function m on X which is holomorphic on M such that $\overline{m^{-1}(0)}$ is reduced and irreducible and $Z = \overline{m^{-1}(0)}$. Let $M' := \text{Kalmod}(M, (Z \cap M), \{q\})$, and let $M'' := M \backslash Z$. Suppose also that $\dim_{\mathbb{C}} X \geq 3$. We also assume that $c_1(M') = c_1(M'') = 0$.

Theorem 2.31. *$SH_*(M'') = SH_*(M')$.*

This theorem follows easily from the key theorem 1.3 and the following theorem:

Theorem 2.32. *There exist compact convex Lefschetz fibrations $E'' \subset E'$ respectively satisfying the conditions of Theorem 1.3 such that E' (resp. E'') is convex deformation equivalent to M' (resp. M'').*

This will be proved in the appendix (8). The basic idea of the proof is to use Lefschetz fibrations defined in an algebraic way.

3. Proof of the main theorem

3.1. **Construction of our exotic Stein manifolds.** First of all, we will construct a Stein manifold K_4 diffeomorphic to $\mathbb{C}^4$. We will then construct Stein manifolds K_n diffeomorphic to $\mathbb{C}^n$ for all $n > 3$ from K_4. Finally using

end connect sums we will construct infinitely many Stein manifolds $(K_n^k)_{k\in\mathbb{N}}$ diffeomorphic to $\mathbb{C}^n$ for all $n > 3$.

We define the polynomial $P(z_0, \dots, z_3) := z_0^7 + z_1^2 + z_2^2 + z_3^2$ and $V := \{P = 0\} \subset \mathbb{C}^4$. Let $\mathbb{S}^7$ be the unit sphere in $\mathbb{C}^4$.

Theorem 3.1. [5] *$V \cap \mathbb{S}^7$ is homeomorphic to $\mathbb{S}^5$.*

Since V is topologically the cone on the link $V \cap \mathbb{S}^7$,

Corollary 3.2. *V is homeomorphic to $\mathbb{R}^6$.*

Let $p \in V \setminus \{0\}$. We let $K_4 := \text{Kalmod}(\mathbb{C}^4, V, \{p\})$. Now by Corollary 3.2 and Lemma 2.27 we have that K_4 is contractible. Hence by Theorem 2.30 we have that K_4 is diffeomorphic to $\mathbb{C}^4$. We will now construct the varieties K_n by induction. Suppose we have constructed the varieties $K_4, \dots, K_n$, we wish to construct the variety K_{n+1}. We do this using example 2.29. This means that we will define $K_{n+1} := \text{Kalmod}(K_n \times \mathbb{C}, K_n \times \{0\}, (q, 0))$ where q is a point in K_n. All these are affine varieties and hence have Stein structures by Example 2.8. Finally, we define

$$K_n^k := \underset{i=1\dots k}{\#_e} K_n$$

which is the k fold end connect sum of K_n. The aim of this paper is to show that if $K_n^k \sim K_n^m$ then $k = m$.

3.2. **Proof of the main theorem in dimension 8.** Here we will prove Theorem 1.1 in dimension 8. In this section we wish to show that if $K_4^k \sim K_4^m$ then $k = m$. Let $M' := K_4^1$. By Theorem 2.20, $SH_*(K_4^k) = \prod_{i=0}^k SH_*(M')$. Hence if $i(M')$ is finite, $i(K_4^k) = i(M')^k$ where $i(M)$ denotes the number of idempotents of $SH_*(M)$ for any Stein manifold M. So in order to distinguish these manifolds, we need to show that $1 < i(M') < \infty$. Let $M'' := \mathbb{C}^4 \setminus V$ where V is defined in section 3.1. By Theorem 2.31, we have that $SH_*(M'') = SH_*(M')$. We have that $1 < i(M'') < \infty$ by Theorem 7.7, hence $1 < i(M') < \infty$.

3.3. **Proof of the theorem in dimensions greater than 8.** Here we will prove Theorem 1.1 in dimensions greater than 8. Let $K_n := K_n^1$. For each $n > 4$ we need to show that $1 < i(K_n) < \infty$ in order to distinguish K_n^k. This is done by induction. Suppose that $1 < i(K_n) < \infty$ for some n, then we wish to show that $1 < i(K_{n+1}) < \infty$. We have by Theorem 2.31, that $SH_*(K_{n+1}) \cong SH_*(K_n \times \mathbb{C}^*)$. Let $B := K_n \times \mathbb{C}^*$. Let $SH_*^{\text{contr}}(\mathbb{C}^*)$ be the subring of $SH_*(\mathbb{C}^*)$ with H_1 grading 0.

One can check that $SH_*^{\text{contr}}(\mathbb{C}^*)$ is a subring isomorphic to $H^{1-*}(\mathbb{C}^*)$. In particular $SH_1^{\text{contr}}(\mathbb{C}^*) \cong \mathbb{Z}/2$. By the Künneth formula (see [19]), we have that

$$SH_{(n+1)+*}(B) \cong SH_{n+*}(K_n) \otimes SH_{1+*}(\mathbb{C}^*).$$

This ring is naturally graded by $H_1(\mathbb{C}^*)$. Hence any idempotent must be an element of

$$SH_{n+*}(K_n) \otimes SH^{\text{contr}}_{1+*}(\mathbb{C}^*) \subset SH_{n+*}(K_n) \otimes SH_{1+*}(\mathbb{C}^*)$$

by Lemma 7.6. The ring $SH^{\text{contr}}_{1+*}(\mathbb{C}^*)$ is naturally graded by the Conley-Zehnder index taken with negative sign because $c_1(\mathbb{C}^*) = 0$. This means that any idempotents must live in:

$$SH_{n+*}(K_n) \otimes SH^{\text{contr}}_1(C^*) \cong SH_{n+*}(K_n) \otimes \mathbb{Z}/2 \cong SH_{n+*}(K_n).$$

Hence $i(K_{n+1}) = i(K_n)$. This means that by induction we have $1 < i(K_n) < \infty$ for all $n > 3$ as we proved $1 < i(K_3) < \infty$ in section 3.2. This proves our theorem.

4. Lefschetz fibration proofs

Here is the statement and proof of Theorem 2.15: *Let (E, π) be a compact convex Lefschetz fibration. There exists a constant $K > 0$ such that for all $k \geq K$ we have: $\omega := \Omega + k\pi^*(\omega_S)$ is a symplectic form, and the ω-dual λ of $\theta := \Theta + k\pi^*\theta_S$ is transverse to ∂E and pointing outwards.*

Proof. We let K be a large constant so that $\omega := \Omega + \pi^*(K\omega_S)$ is a symplectic form (see [22, Lemma 1.5]). Let $\theta'_S := K\theta_S$ and $\omega'_S = d\theta'_S$ and λ'_S be the ω'_S-dual of θ_S. Let $U \times V$ be some trivialisation of π around some point $p \in \pi^{-1}(\partial S)$ where $U \subset F$ and $V \subset S$. We let V be some small half disk around $\pi(p)$ and U is some small open ball. Let $\pi^1 : U \times V \twoheadrightarrow U$ be the natural projection. Let λ_F be the $\Omega|_F$-dual of $\Theta|_F$, and λ_Q be the horizontal lift of λ'_S. The ω-dual of Θ is equal to:

$$\lambda_F + W$$

where W is ω-orthogonal to the vertical plane field tangent to the fibres and is equal to 0 near the horizontal boundary of E. The ω-dual of $K\pi^*\theta'_S$ is equal to:

$$G\lambda_Q$$

where G is some function on $U \times V$. This means that the ω-dual of θ is:

$$\lambda = \lambda_F + W + G\lambda_Q.$$

Because $W = 0$ near the horizontal boundary and because the horizontal subspaces are tangent to the horizontal boundary, we have that λ is transverse to the horizontal boundary. In order to show that λ is transverse to the vertical boundary we need to ensure that we can make G very large compared to $\lambda_F + W$. This can be done by making K sufficiently large. □

5. A cofinal family compatible with a Lefschetz fibration

In this section we construct a family of Hamiltonians $H_k : \widehat{E} \to \mathbb{R}$ which behave well with respect to the Lefschetz fibration, so that

$$SH^l_*(E) := \varinjlim_k SH_*(E, H_k, J) = SH_*(E).$$

This would be obvious if H_k belonged to the "usual" class (i.e. linear of slope k on the cylindrical end) but it is not obvious that our Hamiltonians are linear with respect to some cylindrical end. Throughout this section, (E, π) is a compact convex Lefschetz fibration. We let $\Theta, \Omega, \theta, \omega$ be defined as in section 2.2.

Theorem 5.1. *Let $H : \widehat{E} \to \mathbb{R}$ be Lefschetz admissible for E with non-degenerate orbits. Then the space of regular almost complex structures $\mathcal{J}_{\mathrm{reg}}(\widehat{E}, H)$ is of second category in the space $\mathcal{J}^h(\widehat{E})$ of admissible almost complex structures with respect to the C^∞ topology.*

With a regular almost complex structure, it is possible to define symplectic homology $SH_*(\widehat{E}, H, J)$ with the pair of pants product $SH_k(\widehat{E}, H, J) \otimes SH_l(\widehat{E}, H, J) \to SH_{l+k-n}(\widehat{E}, 2H, J)$. Regular almost complex structures are almost complex structures such that some natural section of a Banach bundle associated with the almost complex structure is transverse to the zero section. This section is described by some linearized version of the perturbed Cauchy-Riemann equations. This theorem comes from using results in [12]. Viterbo in [29, section 1.1] shows us why the Theorem is true in the context of open symplectic manifolds where the almost complex structure is fixed outside a large compact set and where there is a maximum principle as in [29, Lemma 1.8] or [18, Lemma 1.5]. This argument applies if we use the maximum principle in 5.2 below. This ensures that the moduli spaces of Floer trajectories are manifolds. For non-generic (H, J), $SH_*(H, J)$ is defined via small perturbations, and is independent of choice of small perturbation via continuation map techniques. We also need a maximum principle to ensure that the Floer moduli spaces have compactifications.

Let W be a connected component of ∂S where S is the base. Now $\widehat{S}$ has a cylindrical end $W \times [0, \infty)$. Let r_S be the coordinate for $[1, \infty)$. Let $u : \mathbb{D} \to \widehat{E}$ satisfy Floer's equations for some $J \in \mathcal{J}^h\widehat{E}$ and some admissible Hamiltonian H. Here $\mathbb{D}$ is the unit disk parameterized by coordinates (s, t). We can write $H = \pi^* H_S + \pi_1^* H_F$ as in Definition 2.21. We assume that $H_F = 0$ on F.

Lemma 5.2. *The function $f := r_S \circ \pi \circ u$ cannot have an interior maximum for r_S large.*

Proof. Let f have an interior maximum at $q \in \mathbb{D}$. Let U be a small neighbourhood of $u(q)$. The symplectic form ω on $\widehat{E}$ splits the tangent space of E into vertical planes and horizontal planes. Let V be the vertical plane field, and let P be the horizontal plane field (the ω-orthogonal of vertical

tangent spaces of π). Let ω_S be the symplectic form on the base S, then $\omega_P := \pi^*\omega_S|_P$ is non-degenerate. This means that there exists a function

$$g : \pi^{-1}(W \times [0,\infty)) \to (0,\infty)$$

such that $g\omega_P = \omega|_P$. We may assume that $J(P) \subset P$ because J is compatible with $\widehat{E}$ if r_S is large. Let p be the natural projection $TE \to P$ induced by the splitting $TE = V \bigoplus P$.

Floer's equation for u splits up into a horizontal part associated to P and a vertical part associated to V. The horizontal part can be expressed as:

$$p(\frac{\partial u}{\partial s}) + Jp(\frac{\partial u}{\partial t}) = -J\frac{1}{g}G$$

where G is a vector field on P which is the ω_P-orthogonal to $d\pi^*H_S|_P$ in P. Hence $u' := \pi \circ u$ satisfies the equation:

$$\frac{\partial u'}{\partial s} + j\frac{\partial u'}{\partial t} = -j\frac{1}{u^*(g)}X_{H_S}$$

where j is the complex structure of S, and X_{H_S} is the Hamiltonian vector field of H_S in S. Rearranging the above equation gives:

$$u^*(g)\frac{\partial u'}{\partial s} + ju^*(g)\frac{\partial u'}{\partial t} = -jX_{H_S}.$$

Now locally around the point q, we can choose a reparameterization of the coordinates (s,t) to new coordinates (s',t') so that u' satisfies:

$$\frac{\partial u'}{\partial s'} + j\frac{\partial u'}{\partial t'} = -jX_{H_S}$$

(i.e. $\frac{\partial s'}{\partial s} = \frac{\partial t'}{\partial t} = \frac{1}{u^*(g)}$ and $\frac{\partial t'}{\partial s} = \frac{\partial s'}{\partial t} = 0$). The above equation is Floer's equation which doesn't have a maximum by [18, Lemma 1.5]. This gives us a contradiction as we assumed f had a maximum at q. □

If H_s is a non-decreasing sequence of Lefschetz admissible Hamiltonians, then we can use the same methods as above to prove a maximum principle for u satisfying the parameterized Floer equations

$$\partial_s u + J_{s,t}(u(s,t))\partial_t u = \nabla^{g_t} H_{s,t}.$$

Note we also have a maximum principle in the vertical direction as well. We have that the region A as defined in Definition 2.21 looks like $\partial F \times [1,\infty) \times \widehat{S}$. Let r_F be the coordinate for $[0,\infty)$ in this product. Let $\pi_1 : A \twoheadrightarrow \partial F \times [1,\infty)$ be the natural projection. If a Floer trajectory u has an interior maximum with respect to r_F for r_F large, then $\pi_1 \circ u$ satisfies Floer's equations on F and hence has no maximum by [18, Lemma 1.5]. This gives us a contradiction. Hence $r_F \circ u$ has no maximum for r_F large. The above maximum principles and the regularity result from Theorem 5.1 ensures that $SH_*(\widehat{E}, H)$ is well defined.

Definition 5.3. *Let M be a manifold with contact form α. Let $S : \{\text{Reeb orbits}\} \to \mathbb{R}$, $S(o) := \int_o \alpha$. Then the* **period spectrum** *$\mathcal{S}(M)$ is the set $\text{im}(S) \subset \mathbb{R}$. We say that the period spectrum is discrete and injective if the map S is injective and the period spectrum is discrete in $\mathbb{R}$.*

Definition 5.4. *Let H be a Hamiltonian on a symplectic manifold M. Then the* **action spectrum** *$\mathcal{S}(H)$ of H is defined to be:*

$$\mathcal{S}(H) := \{A_H(o) : o \text{ is a 1-periodic orbit of } X_H\}.$$

A_H is the action defined in section 2.3.

We let F be a smooth fibre of (E, π) and $\Theta_F := \Theta|_F$. Also we let S be the base of this fibration. Let r_S and r_F be the "cylindrical" coordinates on $\hat{S}$ and $\hat{F}$ respectively (i.e. $\omega_S = d(r_S\theta_S)$ on the cylindrical end at infinity and similarly with r_F). Let W be some connected component of the boundary of S. Let $C := \pi^{-1}(W) \times [1, \infty)$. Note: we will sometimes write r_S instead of $\pi^* r_S$ so that calculations are not so cluttered. We hope that this will make things easier to understand for the reader.

The boundary of E is a union of 2 manifolds whose boundaries meet at a codimension 2 corner. We can smooth out this corner so that E becomes a compact convex symplectic manifold M such that the completion $\widehat{M}$ is exact symplectomorphic to $\widehat{E}$. This means we can view M as an exact submanifold of $\widehat{E}$. We will let $\partial M \times [1, \infty)$ be the cylindrical end of $\widehat{E} = \widehat{M}$ and we will let r be the coordinate for the interval $[1, \infty)$. We will assume that the period spectrum of ∂M is discrete and injective. Let $\varrho_p : \widehat{E} \to \mathbb{R}$ be an admissible Hamiltonian on $\widehat{M} = \widehat{E}$ with slope p with respect to the cylindrical end $\partial M \times [1, \infty)$ where p is a positive integer. We will also assume that $\varrho_p < 0$ inside M and that ϱ_p tends to 0 in the C^2 norm inside M as p tends to infinity, and that $\varrho_p = h_p(r)$ in the cylindrical end. We assume that $h_p'(r) \geq 0$ for all r and $h_p'(r) = p$ for $r \geq 2$. We also assume that $h_p''(r) \geq 0$ for all r. We can perturb the boundary of M to ensure that no positive integer is in the period spectrum of ∂M and hence p is not in the action spectrum. Hence the family $(\varrho_p)_{p \in \mathbb{N}_+}$ is a cofinal family of admissible Hamiltonians.

Theorem 5.5. *There is a cofinal family of Lefschetz admissible Hamiltonians $K_p : \widehat{E} \to \mathbb{R}$ and a family of almost complex structures $J_p \in \mathcal{J}_{\text{reg}}(\widehat{E}, K_p)$ such that for $p \gg 0$:*

1. *The periodic orbits of K_p of positive action are in 1-1 correspondence with the periodic orbits of ϱ_p. This correspondence preserves index. Also the moduli spaces of Floer trajectories are canonically isomorphic between respective orbits.*
2. *$K_p < 0$ on $E \subset \widehat{E}$.*
3. *$K_p|_E$ tends to 0 in the C^2 norm on E as p tends to infinity.*

This theorem implies that:

$$\varinjlim_p SH_*^{[0,\infty)}(K_p) = \varinjlim_p SH_*(\varrho_p) \tag{1}$$

$SH_*^{[0,\infty)}(K_p) := SH_*(K_p)/SH_*^{(-\infty,0)}(K_p)$ where $SH_*^{(-\infty,0)}$ is the symplectic homology group generated by orbits of negative action. We also have:

$$\varinjlim_p SH_*(K_p) = \varinjlim_p SH_*^{[0,\infty)}(K_p) \tag{2}$$

This is because there exists a cofinal family of Lefschetz admissible Hamiltonians G_p such that:

(1) $G_p < 0$ on $E \subset \widehat{E}$.
(2) $G_p|_E$ tends to 0 in the C^2 norm on E as p tends to infinity.
(3) All the periodic orbits of G_p have positive action.

Property (3) of G_p will follow from Lemma 5.6. Using the fact that both K_p and G_p are cofinal, tending to 0 in the C^2 norm on E and are non-positive on E, there exist sequences p_i and q_i such that:

$$K_{p_i} \leq G_{q_i} \leq K_{p_{i+1}}$$

for all i. Hence:

$$\varinjlim_p SH_*^{[0,\infty)}(G_p) = \varinjlim_p SH_*^{[0,\infty)}(K_p).$$

Property (3) of G_p implies:

$$\varinjlim_p SH_*^{[0,\infty)}(G_p) = \varinjlim_p SH_*(G_p).$$

This gives us equation (2). Combining this with equation (1) gives:

$$\varinjlim_p SH_*(K_p) = \varinjlim_p SH_*(\varrho_p).$$

This proves Theorem 2.22.

Before we prove Theorem 5.5, we need two preliminary Lemmas. We need a preliminary Lemma telling us something about the flow of a Lefschetz admissible Hamiltonian. We let $H = \pi^* H_S + \pi_1^* H_F$ be as in Definition 2.21. We assume that the slope of H_S and H_F is strictly less than some constant $B > 0$. We set H_F to be zero in F, and H_F to be equal to $h_F(r_F)$ in the region $r_F \geq 1$ such that $h_F'(r_F) \geq 0$ and $h_F''(r_F) \geq 0$. We also assume that for some very small $\epsilon > 0$, h_F' is constant for $r_F > \epsilon$ and not in the period spectrum of ∂F so that all the orbits lie in the region $r_F \leq \epsilon$. We define H_S in exactly the same way so that it is zero in S and equal to $h_S(r_S)$ on the cylindrical end of $\widehat{S}$ where h_S has the same properties as h_F. The action of an orbit of H_F in the cylinder $r_F \geq 1$ is $r_F h_F'(r_F) - h_F(r_F)$ and similarly the action of an orbit of H_S in $r_S \geq 1$ is $r_S h_S'(r_S) - h_S(r_S)$, so we can choose ϵ small enough so that the actions of the orbits lie in the interval $[0, B]$ because the slope of H_S and H_F is less than B. We have from

Section 4, $\theta = \Theta + k\pi^*\theta_S$ where Θ is the 1-form associated to the Lefschetz fibration (it is a 1-form such that $\Theta|_F$ makes each fibre F into a compact convex symplectic manifold. Also θ_S is the 1-form making the base S into a compact convex symplectic manifold. The constant k is some large constant.

Lemma 5.6. *For k large enough, there exists a constant Ξ depending only on E and θ (not on H) such that the action of any orbit of H is contained in the interval $[0, \Xi B]$.*

Proof. Inside E, we have that the Hamiltonian is 0 so all the orbits have action 0 there. In the region A as defined in Definition 2.21, we have that the orbits come in pairs (γ, Γ) where γ is an orbit from H_S and Γ is an orbit of positive action from H_F. The action of (γ, Γ) is the sum of the actions of γ and Γ. Both these actions are positive. Also their actions are bounded above by B.

So we only need to consider orbits outside the region $A \cup E$. The Hamiltonian $\pi_1^* H_F$ is zero in this region so we only need to consider $\pi^* H_S$. We will consider the orbits of $\pi^* H_S$ in the region $r_S \geq 1$. In this region, there are no singular fibres of the Lefschetz fibration, so we have a well defined plane field P which is the ω-orthogonal plane field to the vertical plane field which is the plane field tangent to the fibres of π. The Hamiltonian flow only depends on $\omega|_P$ and not the vertical plane field because $\pi^* H_S$ restricts to zero on the vertical plane field. The symplecic form $\omega|_P$ is equal to $Gk\pi^* d\theta_S|_P$ for some function $G > 0$. This means that the Hamiltonian vector field associated to $\pi^* H_S$ is $\frac{1}{G}$ times the horizontal lift of the Hamiltonian vector field associated to H_S in S. Let V be this horizontal lift. The construction of the completion of a Lefschetz fibration before Definition 2.16 ensures that the region $r_S \gg 1$ is a product $W \times [1, \infty)$ where r_S parameterizes the second factor of this product and Θ is a pullback of a 1-form on W via the natural projection $W \times [1, \infty) \to W$. This means that Θ is invariant under translations in the r_S direction (i.e. under the flow of the vector field $\frac{\partial}{\partial r_S}$ which is $\frac{1}{r_S}$ times the horizontal lift of λ_S where λ_S is the Liouville flow in $\widehat{S}$). We also have that $d\theta_S$ is invariant under translations in the r_S direction (i.e. under the flow of $\frac{1}{r_S}\lambda_S$). Hence the symplectic structure ω is also invariant under translations in the r_S direction for $r_S \gg 0$. This means that the function G is bounded above and below by positive constants as the symplectic structure is invariant under translations in the r_S direction and if we travel to infinity in the fibrewise direction (i.e. if we travel into the region A), then $G = 1$. We want bounds on the function $V(\theta)$ because the function G is bounded. Let Y be the Hamiltonian flow of r_S in $\widehat{S}$ and let $\tilde{Y}$ be its horizontal lift to P. We have that $Y(\theta_S) = 1$. This means that $\tilde{Y}(\pi^*\theta_S) = 1$. We also have that $\tilde{Y}(\Theta)$ is bounded because Θ is invariant in the r_S direction for r_S large and $\tilde{Y}(\Theta) = 0$ if we are near infinity in the fibrewise direction. We choose the constant k large enough so that $\tilde{Y}(\theta) = \tilde{Y}(\Theta) + k\tilde{Y}(\pi^*\theta_S) > 0$. This function is also bounded above

because $\tilde{Y}(\Theta)$ is bounded and $k\tilde{Y}(\pi^*\theta_S) = kY(\theta_S) = k$. This choice of k only depends on the Lefschetz fibration and not on H. Now, $V = h'_S(r_S)\tilde{Y}$. Because h'_S bounded below by 0, we have that $V(\theta)$ is bounded below by 0 and bounded above by some constant multiplied by the slope of H_S. All the orbits of H lie in some compact set where H is C^0 small, so the action of an orbit is near $\int_o V(\theta)dx$ where the integral is taken over an orbit o and dx is the volume form on o giving it a volume of 1. This means that the action of these orbits is in the interval $[0, \Xi B]$ for some constant Ξ. This completes our theorem. □

The manifold $\widehat{M} = \widehat{E}$ has a cylindrical end $\partial M \times [1, \infty)$. We let r be the radial coordinate of this cylindrical end. The we define set $\{r \leq R\}$ to be equal to $M \cup (\partial M \times [1, R])$. We define the sets $\{r_F \leq R\}$ and $\{r_S \leq R\}$ in a similar way.

Lemma 5.7. *There exists a constant $\varpi > 0$ such that for all $R \geq 1$, we have that $\{r \leq R\} \subset \{r_S \leq \varpi R\}$ and $\{r \leq R\} \subset \{r_F \leq \varpi R\}$.*

Proof. We will deal with r_S first. The level set $r = R$ is equal to the flow of ∂M along the Liouville vector field λ for a time $\log(R)$. Hence, all we need to do is show that $dr_S(\lambda)$ is bounded above by $e^{\varpi} r_S$. This means that if p is a point in ∂M, then the rate at which $r_S(p)$ increases as we flow p along λ is bounded above by $e^{\varpi} r_S(p)$. Hence if we flow p for a time $\log(R)$ to a point q, then $r_S(q) \leq \varpi R$ which is our result.

We will now show $dr_S(\lambda)$ is bounded above by $e^{\varpi} r_S$ to finish the first part of our proof. We let Θ be a 1-form associated to E as constructed before Definition 2.16. Then $\theta = \Theta + \pi^*\theta_S$ where θ_S is a convex symplectic structure for the base $\widehat{S}$. We have that $\omega = d\Theta + \pi^* d\theta_S$. The construction before Definition 2.16 ensures that the region $r_S \gg 1$ is a product $W \times [1, \infty)$ where r_S parameterizes the second factor of this product and Θ is a pullback of a 1-form on W via the natural projection $W \times [1, \infty) \to W$. This means that Θ is invariant under translations in the r_S direction. Hence $d\Theta$ is also invariant under these translations. Also $\pi^* d\theta_S$ is invariant under translations in the r_S direction. All of this means that the vector field V defined as the ω-dual of Θ is invariant under these translations for r_S large. This implies that $dr_S(V)$ is bounded.

Let V' be the ω-dual of $\pi^*\theta_S$. Let λ_S be the Liouville vector field in $\widehat{S}$. Then $V' = GL$ where L is the horizontal lift of λ_S and $G : \widehat{E} \to \mathbb{R}$ is defined in the proof of Lemma 5.6. The proof of Lemma 5.6, tells us that G is a bounded function. Also, $dr_S(\lambda_S) = r_S$, hence

$$dr_S(V') = G dr_S(L) = G dr_S(\lambda_S) = G r_S$$

Hence $dr_S(V')$ is bounded above by some constant multiplied by r_S. Finally, we have that $\lambda = V + V'$ which means that there exists a $\varpi > 0$ such that $dr_S(\lambda)$ is bounded above by $e^{\varpi} r_S$.

We will now deal with r_F. This is slightly more straightforward because the Lefschetz fibration is a product $\partial F \times [1,\infty) \times \widehat{S}$ and θ splits up in this product as $\Theta + \pi^*\theta_S$, where we can view Θ as 1-form on $\partial F \times [1,\infty)$. We need to bound $dr_F(\lambda)$. In this case, because everything splits in this product, we have that $dr_F(\lambda) = dr_F(\Lambda)$ where Λ is the ω-dual of Θ. This is equal to $r_F \leq e^{\varpi} r_F$ as $\varpi > 0$. Hence we have that $r \leq R$ implies that $r_F \leq \varpi R$. $\square$

Proof. of Theorem 5.5. Let ϱ_p be the Hamiltonian as above. We will write $\varrho = \varrho_p$ for simplicity. The idea of the proof is to modify the Hamiltonian ϱ outside some large compact set so that it becomes Lefschetz admissible and in the process only create orbits of negative action without changing the orbits of ϱ or the Floer trajectories connecting orbits of ϱ. We will do this in three sections. In section (a), we will modify ϱ to a Hamiltonian ς so that it becomes constant outside a large compact set κ while only adding orbits of negative action. This is exactly the same as the construction due to Hermann [15]. In section (b) we will consider a Lefschetz admissible Hamiltonian L which is 0 in the region κ, but has action bounded above so that the orbits of $L + \varsigma$ outside κ have negative action. We define our cofinal family $K_p := L + \varsigma$. (c) we ensure that the Floer trajectories and pairs of pants satisfying Floer's equation connecting orbits of positive action stay inside the region $r \leq 2$.

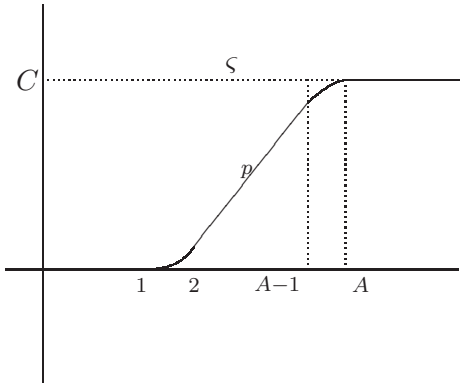
(a) We have that p is the slope of the Hamiltonian ϱ and this is not in the period spectrum of ∂M. Hence, we define $\mu := \mu(p) > 0$ to be smaller than the distance between p and the action spectrum. Define:

$$A = A(p) := 3p/\mu > 1.$$

We can assume that $A > 4$ because we can choose μ to be arbitrarily small. Remember that $\widehat{E} = \widehat{M}$ where M is a compact convex symplectic manifold, and that r is the radial coordinate for the cylindrical end of $\widehat{M}$. We define ς to be equal to ϱ on $r \leq A-1$. On the region $r \geq 1$, we have that ϱ is equal to $h_p(r)$. We will just write h instead of h_p. Set $\varsigma = k(r)$ for $r \geq 1$ with non negative derivative. This means that in the region $1 \leq r \leq A-1$ we have that $h(r) = k(r)$. Hence in $r \leq A-1$ we have that $k''(r) \geq 0$ and $k'(r) \geq 0$, and in the region $2 \leq r \leq A-1$ we have $k'(r) = p$. Also we have that ς is C^2 small and negative for r near 1. Because ς is C^2 small, we can also assume that p is large enough so that for r near 1, $k' \ll p$. Because ϱ_p is cofinal, we can assume that p is large enough so that $h(2) = k(2) > 0$. Both these previous facts mean that $p(A-2) < k(A-1) < p(A-1)$. Outside this region, we define k to be a function with the following constraints: For $r \geq A$ set $k(r)$ to be constant and equal to C where $C = p(A-1)$. In the

region $A-1 \leq r$, $k'' \leq 0$. We assume that $k' \geq 0$ for all $r \geq 1$. Here is a picture:

Figure 5.8.



We want to show that the additional orbits of ς only have negative action. All these orbits lie in the region $r \geq 2$. In fact because p is not in the action spectrum, they lie in the region $r \geq A-1$. In the region $\{r : p-\mu < k'(r) \leq p\}$, we have that ς has no periodic orbits. Also, the action of a periodic orbit is $k'(r)r - k(r)$. Combining these two facts implies that the action of a periodic orbit in the region $2 \leq r$ is less than

$$(p-\mu)r - k(r) \leq (p-\mu)A - p(A-2)$$

$$= -\mu A + 2p = -\mu\frac{3p}{\mu} + 2p = -p < 0$$

Hence we have a Hamiltonian ς equal to ϱ in the region $r \leq 2$ and such that it is constant and equal to $C = p(A-1)$ in the region $r \geq A-1$ and such that all the additional periodic orbits created have negative action.

(b) Lemma 5.6 tells us that there exists a cofinal family of Lefschetz admissible Hamiltonians Λ_p such that the action spectrum of Λ_p is bounded above by some constant Ξ multiplied by the slope of λ_p. We can assume that both the slopes of λ_p are equal to $\sqrt{(}p)$ (if $\sqrt{p}$ is in the action spectrum of the fibre or the base, then we perturb this value slightly to ensure that Λ_p has orbits in a compact set). This means that the action of Λ_p is bounded above by $\Xi\sqrt{p}$. The Hamiltonian Λ_p is equal to zero in E. We will now define a Hamiltonian L_p as follows: We let ϖ be defined as in Lemma 5.7. Set $L_p = 0$ in the region $\{r_S \leq \varpi A\} \cap \{r_F \leq \varpi A\}$. In the region $\{r_S \geq 1\} \cup \{r_F \geq 1\}$, we have that Λ_p is a function of the form $\pi_1^* h_F(r_F) + \pi^* h_S(r_S)$. Here, π_1 is the natural projection: $\partial F \times [1,\infty) \times \widehat{S} \to \partial F \times [1,\infty)$ (this is the same as the projection defined just before Definition 2.21). So, we set the function $\pi_1^* h_F(r_F)$ to be zero outside the domain of definition of π_1. Also, $\pi^* h_S$ is zero outside the region $r_S \geq 1$. We define L_p to be

$$\pi_1^* h_F(r_F - \varpi A) + \pi^* h_S(r_S - \varpi A)$$

in the region $\{r_S \geq \varpi A\} \cup \{r_F \geq \varpi A\}$. Hence we have a well defined function L_p. Because L_p has scaled up, we have that the action spectrum of L_p is equal to ϖA multiplied by the action spectrum of Λ_p. Hence, we have that the action spectrum of L_p is bounded above by $\varpi A \Xi \sqrt{p}$.

Because $\{r \leq A\} \subset \{r_S \leq A\} \cap \{r_F \leq A\}$, we can add L_p to ς without changing the orbits of ς in the region $r \leq A$. Also, the action of the orbits of $\varsigma + L_p$ in the region $r \geq A$ is bounded above by $\varpi A \Xi \sqrt{p} - p(A-1)$. So for p large enough we have that the additional orbits added are of negative action.

(c) We choose an almost complex structure $J \in \mathcal{J}^h(\widehat{E})$ such that on some neighbourhood of the hypersurface $r = 2$, J is admissible. Then [3, Lemma 7.2] and the comment after this Lemma ensure that no Floer trajectory or pair of pants satisfying Floer's equation connecting orbits inside $r < 2$ can escape $r \leq 2$. Hence our Hamiltonian $K_p := \varsigma + L_p$ has all the required properties.

5.1. **A better cofinal family for the Lefschetz fibration.** In this section we will prove Theorem 2.24. We consider a compact convex Lefschetz fibration (E, π) fibred over the disc $\mathbb{D}$. Basically the cofinal family is such that $H_F = 0$. This means that the boundary of F does not contribute to symplectic homology of the Lefschetz fibration. The key idea is that near the boundary of F the Lefschetz fibration looks like a product $\mathbb{D} \times \mathrm{nhd}(\partial F)$ and because symplectic homology of the disc is 0 we should get that the boundary contributes nothing. Statement of Theorem 2.24:

$$SH_*(E) \cong SH_*^{\mathrm{lef}}(E).$$

We will define $F, S(= \mathbb{D}), r_S, r_F, \pi_1$ as in the previous section. This means that the compact convex sympectic manifold F is a fibre of E and S is the base which in this section is equal to $\mathbb{D}$. We also have that r_S is a radial coordinate for the cylindrical end of $\widehat{S}$ which we also identify with $\pi^* r_S$. The map π_1 is the natural projection $(\partial F \times [1, \infty)) \times \widehat{S} \twoheadrightarrow (\partial F \times [1, \infty))$ where $(\partial F \times [1, \infty)) \times \widehat{F}$ is a subset of $\widehat{E}$. The function r_F is a radial coordinate for the cylindrical end of $\widehat{F}$ which we also identify with $\pi_1^* r_F$. Before we prove Theorem 2.24, we will write a short lemma on the $\mathbb{Z}$ grading of $SH_*(E)$.

Lemma 5.9. *Let $\widehat{F} := \pi^{-1}(a) \subset \widehat{E}$ ($a \in \mathbb{D}$). Suppose we have trivialisations of $\mathcal{K}_{\widehat{E}}$ and $\mathcal{K}_{\widehat{S}}$ (these are the canonical bundles for $\widehat{E}$ and $\widehat{S}$ respectively); these naturally induce a trivialisation of $\mathcal{K}_{\widehat{F}}$ away from F. If we smoothly move a, then this smoothly changes the trivialisation.*

Proof. of Lemma 5.9.

We choose a $J \in \mathcal{J}^h(E)$. The bundle E away from E^{crit} has a connection induced by the symplectic structure. Let $A \subset \widehat{E}$ be defined as in Definition 2.21. Let U be a subset of A where

(1) π is J holomorphic.

(2) U is of the form $r \geq K$ where r is the coordinate for $[1, \infty)$ in A (see Definition 2.21).

This means that in U, we have that the horizontal plane bundle $\mathbb{H}$ is J holomorphic. Choose a global holomorphic section of $\mathcal{K}_{\widehat{S}}$ and lift this to a section s of $\mathbb{H}$. Choose a global holomorphic section t of $\mathcal{K}_{\widehat{E}}$. The tangent bundle of $\widehat{F}$ is isomorphic to the ω-orthogonal bundle T of $\mathbb{H}$. This is also a holomorphic bundle. Let $\Lambda^k T$ be the highest exterior power of T. There exists a unique holomorphic section w of $\Lambda^k T$ such that $s \wedge w = t$. Hence, w is our nontrivial holomorphic section of T in $U \cup \widehat{F}$. This can be extended to $A \cup \widehat{F}$ by property (2). □

In the following proof, whenever we talk about indices of orbits of $\widehat{F}$ outside F, we do this with respect to the trivialisation of Lemma 5.9 above. We do not deal with orbits inside F so this trivialisation is sufficient.

Proof. of Theorem 2.24. We start by defining a cofinal family of Lefschetz admissible Hamiltonians $H^\lambda = \pi^* H_S^\lambda + \pi_1^* H_F^\lambda$. To avoid cluttered notation, we suppress the λ and just write H, H_S, H_F instead of $H^\lambda, H_S^\lambda, H_F^\lambda$ unless we need to explicitly deal with λ. We assume that the period spectrum of ∂F is discrete and injective and also that the Reeb orbits of ∂F are non-degenerate. We assume that $H_F = 0$ on F and is equal to $h_F(r_F)$ outside F with $h_F'(r_F) > 0$ and $h_F''(r_F) \geq 0$ when $r_F > 1$. The orbits of H_F consist of constant orbits in F and $\mathbb{S}^1$ families of orbits corresponding to periodic Reeb orbits outside F. We can perturb H_F by a very small amount outside F so that each $\mathbb{S}^1$ family of Reeb orbits becomes a pair of non-degenerate orbits (see [18, Section 3.3]). Hence, we have a Hamiltonian H_F which is equal to 0 inside F and all its orbits outside F are non-degenerate. We set the slope of H_F at infinity to be equal to $\lambda \notin \mathcal{S}(\partial F)$. The completion $\widehat{\mathbb{D}}$ of the disc $\mathbb{D}$ is symplectomorphic to $\mathbb{C}$ with the standard symplectic structure. We have that $r_S(z) = |z|^2$ where $z \in \mathbb{C}$. The function r_S is defined only on the cylindrical end, but we will extend it to the interior of $\mathbb{D}$ by the function $|z|^2$. We set $H_S = \kappa(\lambda) r_S$ on $\mathbb{C}$ where $\kappa(\lambda)$ is some function of the slope λ of H_F such that it is never a multiple of π. Here H_S has exactly one periodic orbit at $0 \in \mathbb{C}$ of index $(2a + 1)$ where a is the integer satisfying $a\pi < \kappa(\lambda) < \pi(a + 1)$ (see [18, Section 3.2]). For each λ, we choose $\kappa(\lambda)$ to be large enough so that the index of the only orbit of H_S is greater than $M + \lambda + 1$ where $M = -\mathrm{ind}(\Gamma)$ such that Γ is an orbit of H_F of lowest index. We also assume that as λ tends to infinity, $\kappa(\lambda)$ also tends to infinity. The Hamiltonians H^λ form a cofinal family of Lefschetz admissible Hamiltonians. Hence for some $J \in \mathcal{J}^h(\widehat{E})$, we have $SH_*(E) = \varinjlim_{\lambda} SH_*(E, H^\lambda, J)$.

We will also define a half Lefschetz admissible Hamiltonian $\bar{H}^\lambda := \pi^* H_S^\lambda + \pi_1^* \bar{H}_F^\lambda$. When appropriate, we will write $\bar{H}, \bar{H}_F$ instead of $\bar{H}^\lambda, \bar{H}_F^\lambda$. We define $\bar{H}_F = 0$ inside F and $\bar{H}_F := \bar{h}_F(r_F)$ in the region $\{r_F \geq 1\}$ where the derivative $\bar{h}_F'$ is so small, that $\bar{H}_F$ has no periodic orbits in the region

$\{r_F > 1\}$. The Hamiltonians $\bar{H}^\lambda$ form a cofinal family of half admissible Hamiltonians, hence $SH_*^{\text{lef}}(E) = \varinjlim_{\lambda} SH_*(E, \bar{H}^\lambda, J)$.

We will construct a natural continuation map $SH_*(\bar{H}^\lambda, J) \to SH_*(H^\lambda, J)$ such that it is an isomorphism in all degrees less than λ. This will prove the theorem for the following reason: Because continuation maps are natural, we can take direct limits with respect to λ, so that we get a map $SH_*(E) \to SH_*^{\text{lef}}(E)$. This map must be an isomorphism because if we choose an integer b, then for $\lambda > b$ we have $SH_b(\bar{H}^\lambda, J) \to SH_b(H^\lambda, J)$ is an isomorphism which implies that $SH_b(E) \to SH_b^{\text{lef}}(E)$ is an isomorphism (because we are taking a direct limit as λ tends to infinity). Hence $SH_*(E) \to SH_*^{\text{lef}}(E)$ is an isomorphism.

We will now show that the continuation map $SH_*(\bar{H}^\lambda, J) \to SH_*(H^\lambda, J)$ is an isomorphism in all degrees less than λ. From now on, we will write $H, H_S, H_F, \bar{H}, \bar{H}_F$ instead of $H^\lambda, H_S^\lambda, H_F^\lambda, \bar{H}^\lambda, \bar{H}_F^\lambda$. The region $\{r_F \geq 1\}$ is a product $([1, \infty) \times \partial F) \times \mathbb{C}$ and the orbits of the Hamiltonian H come in pairs (γ, Γ), where γ is the orbit of H_S and Γ is a non-constant orbit of H_F. The index of this orbit is the sum $\text{ind}(\gamma) + \text{ind}(\Gamma)$, hence its index is greater than $\lambda + 1$. This means that all the orbits of index $\leq \lambda + 1$ are disjoint from the region $\{r_F \geq 1\}$. We have that H_F is not quite a function of r_F in the region $\{r_F \geq 1\}$, as we perturbed it so that it had non-degenerate orbits. Having said that we can assume that for some $\delta > 0$, we have $H_F = h_F(r_F)$ in the region $\{1 \leq r_F \leq 1 + \delta\}$. We also assume that the Lefschetz admissible almost complex structure J is of the form $j + J_F$ in the region $\{r_F \geq 1\}$ viewed as a subset of $\widehat{E}$, where j is an complex structure on $\mathbb{C}$ and J_F is convex on the cylindrical end of $\widehat{F}$. Any Floer trajectory satisfying Floer's equation with respect to (H, J) connecting orbits outside the region $\{r_F > 1\}$, must stay outside this region for the following reason: If $u : S \to \widehat{E}$ is such a curve, then let $\bar{S} := u^{-1}(\{r_F > 1\})$. We can project $u|_{\bar{S}} : \bar{S} \to \{r_F > 1\}$ down from $\{r_F > 1\}$ to $(1, \infty) \times \partial F$. Lemma 7.2 from [3] then tells us that this curve cannot exist. Hence $\bar{S}$ is empty and the claim follows. A similar argument shows that if we had a pair of pants satisfying Floer's equation with respect to Hamiltonians similar to H connecting orbits outside $\{r_F \geq 1\}$, then the curve must also be disjoint from $\{r_F \geq 1\}$. Similarly any Floer trajectory or pair of pants satisfying Floer type equations with respect to $(\bar{H}, J)$ must be disjoint from $\{r_F \geq 1\}$ because all orbits of $\bar{H}$ are disjoint from this region. Let $K_s : \widehat{F} \to \mathbb{R}$ be a monotone increasing sequence of admissible Hamiltonians joining $\bar{H}_F$ and H_F such that in the region $\{1 \leq r_F \leq 1 + \delta\}$, we have that K_s is a function of r_F only. Again if we have a trajectory satisfying the Floer continuation equations with respect to (K_s, J) joining orbits of $\bar{H}$ and H which are disjoint from $\{r_F \geq 1\}$, then by [3, Lemma 7.2] we have that this trajectory is also disjoint from $\{r_F \geq 1\}$. Combining all these facts, we get that the continuation map induced by (K_s, J) is an isomorphism in all

degrees less than λ because $\bar{H} = H$ outside the region $\{r_F \geq 1\}$ and all orbits of H and $\bar{H}$ of degree less than $\lambda + 1$ along with all Floer trajectories and pairs of pants connecting them are disjoint from this region. This completes the proof of Theorem 2.24. □

6. $SH^{\mathrm{lef}}_*(\widehat{E})$ and the Kaliman modification

In this section we prove Theorem 2.25. Throughout this section we assume that E' and E'' are Lefschetz fibrations as described in section 2.4. We recall the situation:

(1) E'' is a subfibration of E' over the same base.
(2) The support of the parallel transport maps of E' are contained in the interior of E''.
(3) There exists a complex structure $J_{F'}$ (coming from a Stein domain) on F' such that any $J_{F'}$-holomorphic curve in F' with boundary in F'' must be contained in F''.

We wish to prove that $SH^{\mathrm{lef}}_*(E') \cong SH^{\mathrm{lef}}_*(E'')$ as rings.

Proof. of Theorem 2.25. Fix $\lambda > 0$. The value λ is going to be the slope of some Hamiltonian, we can always perturb λ slightly so that it isn't in the action spectrum of the boundary. By Theorem 9.1 we can choose an almost complex structure $J_{F',1}$ on $\widehat{F'}$ after a convex deformation away from F' such that it is convex with respect to some cylindrical end at infinity and such that any $J_{F',1}$-holomorphic curve in F' with boundary in F'' must be contained in F''. The reason is because we can ensure that $J_{F',1} = J_{F'}$ in $F' \subset \widehat{F'}$ and that any $J_{F',1}$-holomorphic curve with boundary in $F'' \subset F'$ is contained in F' by Theorem 9.1 hence is contained in F'' by property (3) above. Supposing we have a Hamiltonian $H_{F'}$ which is of the form $h_{F'}(r_{F'})$ on the cylindrical end where $r_{F'}$ is the radial coordinate and $h' \geq 0$ and $H_{F'} = 0$ elsewhere. Then, any curve (Floer cylinder or pair of pants) with boundary in F'' satisfying Floer's equations with respect to $H_{F'}$ and $J_{F',1}$ must be contained in F''. We choose h' small enough so that $H_{F'}$ has no periodic orbits in the region $r_{F'} > 1$. The convex deformation mentioned in Theorem 9.1 fixes $F' \subset \widehat{F'}$ hence it induces a convex deformation on $\widehat{E}$. This is because the region where we deform $\widehat{E}$ looks like a product $\mathbb{C} \times (\widehat{F'} \setminus F')$. From now on we assume that the fibres of $\widehat{E}$ have this almost complex structure $J_{F',1}$ with this cylindrical end.

A neighbourhood of $\partial F''$ in F' is symplectomorphic to $L := (-\epsilon, \epsilon) \times \partial F''$ with the symplectic form $d(r\alpha'')$. Here, r is a coordinate in $(-\epsilon, \epsilon)$ and α'' is the contact form for $\partial F''$. We also choose ϵ small enough so that L is disjoint from the support of the parallel transport maps in F'. Let $\bar{F}'' := F'' \setminus ((-\epsilon/3, 0] \times \partial F'')$. We can choose an almost complex structure $J' \in \mathcal{J}(\widehat{F'})$ with the following properties:

(1) There exists a $\delta > 0$ such that any holomorphic curve meeting both boundaries of $[-\epsilon, -\epsilon/2] \times \partial F''$ has area greater than δ. This is true by the monotonicity lemma [19, Lemma 1].
(2) $J' = J_{F',1}$ on $\widehat{F'} \setminus \bar{F}''$. This means that any curve (cylinder or pair of pants) satisfying Floer's equations with respect to $H_{F'}$ and J' with boundary in F'' is contained entirely in F''.

Construct an almost complex structure J on $\widehat{E'}$ as follows: The parallel transport maps on $\widehat{F'} \setminus \bar{F}''$ are trivial, hence there is a region W of $\widehat{E'}$ symplectomorphic to $\mathbb{C} \times (\widehat{F'} \setminus \bar{F}'')$. We set $J|_W$ to be the product almost complex structure $J_{\mathbb{C}} \times J'$ where $J_{\mathbb{C}}$ is the standard complex structure on $\mathbb{C}$. We then extend $J|_W$ to some J compatible with the symplectic form ω' such that π' is J-holomorphic outside some large compact set. Let H be a Hamiltonian of the form $\pi^* K + \pi_1^* H_{F'}$ where K is admissible of slope λ on the base $\mathbb{C}$ and $\pi_1 : W \twoheadrightarrow \widehat{F'} \setminus \bar{F}''$ is the natural projection map. J has the following properties:

(1) Any curve u satisfying Floer's equations with respect to H meeting both boundaries of $([-\epsilon, -\epsilon/2] \times \partial F'') \times \mathbb{C} \subset \widehat{E'}$ must have energy $\geq \delta$. (u can be a cylinder or a pair of pants).
(2) Any such u connecting orbits inside E'' must be entirely contained in E''.

Property (2) is true because: Let u be a curve satisfying Floer's equations connecting orbits in E'', then composing $u|_{u^{-1}(W)}$ with the natural projection $W \twoheadrightarrow \widehat{F'} \setminus \bar{F}''$ gives us a curve w with boundary in F''. This means that w is contained in F'', and hence u is contained in E''. Also if u meets both boundaries of $([-\epsilon, -\epsilon/2] \times \partial F'') \times \mathbb{C} \subset \widehat{E'}$, then the projected curve w has energy $\geq \delta$ which means that u has energy $\geq \delta$. Hence Property (1) is true.

We perturb $H_{F'} : \widehat{F'} \to \mathbb{R}$ slightly so that:

(1) It is equal to 0 in F''.
(2) The only periodic orbits of $H_{F'}$ are constant orbits.
(3) The action spectrum of $H|_{\widehat{F'} \setminus F''}$ is discrete and injective and contained in $(-\delta/4, \delta/4)$.
(4) We leave $H_{F'}$ alone on the cylindrical end.
(5) All the orbits in $\widehat{F'} \setminus F''$ are of negative action and non-degenerate.
(6) we can ensure that $H_{F'}$ has very small positive slope with respect to the cylindrical end of F'' on the region $(\epsilon/2, \epsilon) \times \partial F'' \subset F'$.

Let $\delta_1 > 0$ be the smallest distance between 0 and the action value of an orbit of $H_{F'}$ of negative action. Here we fix some integer $m > 0$. We can assume that the critical points of our Lefschetz fibration in $\mathbb{C}$ form a regular polygon with centre the origin. Draw a straight line from the origin to each critical point and let G be the union of these lines. Let $X := \frac{r}{2} \frac{\partial}{\partial r}$ be an outward pointing Liouville flow. We choose a loop l around G so that the disc V with $\partial V = l$ has volume v where v can be chosen arbitrarily small,

and such that X is transverse to this loop. This forms a new cylindrical end ϱ for $\mathbb{C}$. Now let H^V_λ be a Hamiltonian on V with slope λ. We assume that H^V_λ has the following properties:

(1) All the orbits are non-degenerate.
(2) The action of any orbit is in the region $[0, 2v\lambda]$.
(3) All orbits of index $\leq m + n$ are exact. ($2n$ is the dimension of our symplectic manifold)
(4) $2v\lambda < \min(\delta/2, \delta_1)$
(5) H^V_λ is constructed in the same way as the Hamiltonian H_S mentioned before Lemma 5.6, but we do this with respect to the new cylindrical end ϱ of $\mathbb{C}$.

We let $K_\lambda = \pi^*(H^V_\lambda) + \pi_1^* H_{F'}$. Let $B := \mathbb{C} \times (\widehat{F'} \setminus F'') \subset \widehat{E'}$. The Hamiltonian K_λ is of the form $\pi^*(H^V_\lambda) + \pi_1^* H_{F''}$ on B. Hence the orbits on B come in pairs (γ, Γ), where γ corresponds to a periodic orbit of H^V_λ and Γ is a constant periodic orbit of $H_{F''}$. The action difference between two orbits is $\leq \delta$ due to property (3) for $H_{F'}$ and properties (2) and (4) for H^V_λ. This implies that any Floer trajectory connecting orbits (γ_1, Γ_1) and (γ_2, Γ_2) inside B must stay inside B, due to property (1) from the properties of J. Also the definition of δ_1 combined with properties (2) and (4) for H^V_λ imply that all orbits inside B have negative action.

Hence we have a subcomplex C^λ_B generated by orbits in B. The Hamiltonian K_λ in the region $\widehat{E} \setminus B$ is equal to $\pi^*(H^V_\lambda)$. Hence, using Lemma 5.6, we have that all the orbits of $\pi^*(H^V_\lambda)$ have non-negative action. This means that we have a quotient complex $C^\lambda_{E''} := C^\lambda_{E'}/C^\lambda_B$ where $C^\lambda_{E'}$ is the complex generated by all orbits. By property (6) of $H_{F'}$ and property (2) of J we have that the limit of $H_*(C^\lambda_{E''})$ as λ tends to infinity is isomorphic to $SH^{\text{lef}}_*(E'')$. So in order to show that $SH^{\text{lef}}_*(E'')$ is isomorphic to $SH^{\text{lef}}_*(E')$ we need to show that the homology of C^λ_B is zero in all degrees $\leq m$. This would imply that $SH^{\text{lef}}_*(E'') = SH^{\text{lef}}_*(E')$ in all degrees $< m$. As λ increases, we can make m increase with λ, and this means that $SH^{\text{lef}}_*(E'') = SH^{\text{lef}}_*(E')$ in all degrees. The point is that all the orbits in C^λ_B have negative action, so any Floer trajectory starting at one of these orbits must also finish at one of these orbits. Also, any Floer trajectory connecting two of these orbits must be contained in B. We have that B is a product. We can assume that it has the product almost complex structure. This means that Floer trajectories between (γ_1, Γ_1) and (γ_2, Γ_2) come in pairs (u, U) where u is a Floer trajectory in $\mathbb{C}$ connecting γ_1 and γ_2 and U is a Floer trajectory connecting Γ_1 and Γ_2. Hence by a Künneth formula, we have that the homology of C^λ_B is isomorphic to $SH_*(H^V_\lambda) \otimes SH^{(-\infty,0)}_*(H_{F'})$, where $SH^{(-\infty,0)}_*(H_{F'})$ is the subcomplex generated by orbits negative action. We have that in all degress $\leq m$, that $SH_*(H^V_\lambda) = 0$, and that $SH^{(-\infty,0)}_*(H_{F'})$ is 0 in negative degrees which means that $SH_*(H^V_\lambda) \otimes SH^{(-\infty,0)}_*(H_{F'})$ is zero in all degrees $\leq m$.

This in turn implies that the homology of C_B^λ is zero in all degrees $\leq m$. This gives us our isomorphism:

$$SH_*^{\text{lef}}(E'') \cong SH_*^{\text{lef}}(E').$$

□

7. Brieskorn Spheres

In this section we will mainly be studying the variety V as constructed in section 3.1 and also the variety $M'' := \mathbb{C}^4 \setminus V$. The variety V is equal to $\{z_0^7 + z_1^2 + z_2^2 + z_3^2 = 0\} \subset \mathbb{C}^4$.

7.1. **Parallel transport.** There is a natural symplectic form on $\mathbb{C}^4$ (induced from an ample line bundle on its compactification $\mathbb{P}^4$). We have a holomorphic map $P := z_0^7 + z_1^2 + z_2^2 + z_3^2$ with one singular point at 0. We can view P as a fibration which is compatible with this symplectic form as in Definition 2.11. These fibrations have a natural connection which is induced from the symplectically orthogonal plane fields to the fibres. We prove:

Theorem 7.1. *Parallel transport maps are well defined for P.*

Proof. We first of all compactify $\mathbb{C}^4$ to $\mathbb{P}^4$. We let P' be a holomorphic section of $E := \mathcal{O}_{\mathbb{P}^4}(7)$:

$$P'([z_0 : \cdots : z_4]) := z_0^7 + z_4^5 z_1^2 + z_4^5 z_2^2 + z_4^5 z_3^2.$$

This is equal to P on the trivialisation $z_4 = 1$. We also have another section Q defined by

$$Q([z_0 : \cdots : z_4]) := z_4^7.$$

The map P can be extended to a rational map $P'' : \mathbb{P}^4 \dashrightarrow \mathbb{P}^1$, where $P'' = \frac{P'}{Q}$. Fix an identification $\mathbb{C}^4 = \mathbb{P}^4 \setminus Q^{-1}(0)$. We now have that

$$P = \frac{P'}{Q}.$$

Let $\|.\|_E$ be a positive curvature metric on the ample bundle E. We have a symplectic structure and Kähler form defined in terms of the plurisubharmonic function

$$\phi = -\log \|Q\|_E^2.$$

In order to show that P has well defined parallel transport maps we need to construct bounds on derivatives similar to the main theorem in [13, section 2]. We take the vector field ∂_z on the base $\mathbb{C}$. It has a unique lift with respect to the Kähler metric which is:

$$\xi := \frac{\nabla P}{\|\nabla P\|^2}.$$

Here $\|.\|$ is the Kähler metric and ∇P is the gradient of P with respect to this metric. Take a point p on $D := \{z_4 = 0\}$. We can assume without loss of generality that this lies in the chart $\{z_1 = 1\}$. In this chart we have that

the metric $\|.\|_E = e^{\sigma}|.|$ where σ is a smooth function and $|.|$ is the standard Euclidean metric with respect to this chart. Then:

$$\phi = -\log \|Q\|^2 = -\log |Q|^2 - \sigma.$$

The notation $\lesssim$ means that one term is less than or equal to some constant times the other term. Hence we get:

$$\begin{aligned} B := \left|\xi.\phi\right| &\leq \frac{|\langle \nabla P, \nabla \sigma\rangle|}{||\nabla P||^2} + \frac{2|Q| \cdot |\langle \nabla Q, \nabla P\rangle|}{||\nabla P||^2 \cdot |Q|^2} \\ &\lesssim \frac{1}{||\nabla P||} + \frac{||\nabla Q||}{||\nabla P|| \cdot |Q|} \end{aligned} \tag{3}$$

We get similar equations to 3 in the other charts $\{z_i = 1\}$. If we can show that for any compact set $T \subset \mathbb{C}$ the function B is bounded above by a constant K in the region $T_1 := P^{-1}(T) \setminus A$ where $A = \{|z_0|, |z_1|, |z_2|, |z_3| \leq 1\}$, then we have well defined parallel transport maps. This is because we get similar bounds if we lift other vectors of unit length (i.e. $c\partial_z$ where $c \in U(1)$). Hence if we have a path, then $\left|\xi.\phi\right|$ is bounded above by a constant on this path. This ensures that the transport maps do not escape to infinity. In the chart $\{z_3 = 1\}$, we have that for $1 \leq i \leq 2$,

$$\partial_i P = 2z_i/z_4^2,$$

$$\partial_0 P = 7z_0^6/z_4^7.$$

We have the following bounds on derivatives:

$$|\nabla Q| \lesssim \frac{|Q|}{|z_4|}.$$

Combining this with equation 3 gives:

$$\begin{aligned} B &\lesssim \frac{1 + |z_4|}{|z_4| \left(\sum_{j=0, j\neq 3}^{4} |\partial_i P|\right)} \\ &\lesssim C := (1 + |z_4|)/ \left[\frac{7|z_0|^6}{|z_4|^6} + \frac{2|z_1|}{|z_4|} + \frac{2|z_2|}{|z_4|} + |\partial_4 P||z_4|\right] \\ &\lesssim (1 + |z_4|)/ \left(|z_0|^6/|z_4|^6 + |z_1|/|z_4| + |z_2|/|z_4|\right). \end{aligned}$$

Hence on the chart $\{z_4 = 1\}$,

$$B \lesssim (1 + |z_3|^{-1})/ \left(|z_0|^6 + |z_1| + |z_2|\right) \tag{4}$$

By symmetry we also have

$$B \lesssim (1 + |z_2|^{-1})/ \left(|z_0|^6 + |z_1| + |z_3|\right) \tag{5}$$

$$B \lesssim (1 + |z_1|^{-1})/ \left(|z_0|^6 + |z_2| + |z_3|\right) \tag{6}$$

In the chart $\{z_0 = 1\}$ we have:

$$B \lesssim \frac{1 + |z_4|}{|z_4| \left(\sum_{j=1}^{4} |\partial_i P|\right)}$$

$$\lesssim (1+|z_4|)/\left[\frac{2|z_1|}{|z_4|}+\frac{2|z_1|}{|z_4|}+\frac{2|z_2|}{|z_4|}+|\partial_4 P||z_4|\right].$$

So:

$$B \lesssim (1+|z_4|)|z_4|/\left(|z_1|+|z_2|+|z_3|\right)$$

so in the chart $\{z_4=1\}$, we get a bound:

$$B \lesssim (1+|z_0|^{-1})/\left(|z_1|+|z_2|+|z_3|\right) \tag{7}$$

Suppose for a contradiction that there is a sequence of vectors $(z_0^i, z_1^i, z_2^i, z_3^i)$ lying in T_1 such that B tends to infinity as i tends to infinity. If (after passing to a subsequence) z_0^i tends to infinity, then equation 7 tells us that z_1^i, z_2^i, z_3^i are all bounded. But this is impossible as $(z_0^i, z_1^i, z_2^i, z_3^i)$ lies in T_1 which means that z_0^i is bounded. Similarly, using equations 4,5,6 we get that z_j^i is bounded. Hence B is bounded away from the compact set $\{|z_0|, |z_1|, |z_2|, |z_3| \leq 1\}$. This means that B is bounded when restricted to T_1, so we have well defined parallel transport maps. □

Let $(\mathbb{C}^4, \theta)$ be the convex symplectic manifold induced by the compactification $\mathbb{C}^4 \hookrightarrow \mathbb{P}^4$. Because parallel transport maps for P are well defined we can use ideas from [23, section 19b] to deform the 1-form θ on $\mathbb{C}^4$ through a series of 1-forms θ_t such that:

(1) each $\omega_t := d\theta_t$ is compatible with P as in Definition 2.11 and θ_t is a convex symplectic deformation on $\mathbb{C}^4$.
(2) The parallel transport maps of P with respect to the connection induced by ω_1 are trivial at infinity. This means that near infinity, P looks like the natural projection $C \times \mathbb{C} \twoheadrightarrow \mathbb{C}$ where C is the complement of some compact set in V.
(3) For a smooth fibre F of P, (F, θ_1) is exact symplectomorphic to (F, θ_0).

We have that $M'' = \mathbb{C}^4 \setminus P^{-1}(0)$, so we can restrict P to a fibration $P'' = P|_{M''} : M'' \to \mathbb{C}^*$. Let θ_S be a convex symplectic structure on $\mathbb{C}^*$ with the property that $\theta_{M'',t} := \theta_t|_{M''} + P''^*\theta_S$ is a convex symplectic structure for M''. Let θ'' be a convex symplectic structure on M'' constructed as in Example 2.8. It is convex deformation equivalent to $(M'', \theta_{M'',0})$ as follows: Let F be a fibre of P'', then $(F, \theta_{M'',0}|_F)$ is convex deformation equivalent to $(F, \theta''|_F)$ by Lemma 2.9 as both convex structures come from Stein structures constructed algebraically as in Example 2.8. This deformation is $(1-t)\theta_{M'',0}|_F + t\theta''|_F$. The following family of 1-forms $\Theta_t := (1-t)\theta_{M'',0} + t\theta''$ induces a convex symplectic deformation (we might have to add $\pi^*\theta'_S$ to $\theta_{M'',0}$ and θ'' where θ_S is a convex symplectic structure on $\mathbb{C}^*$ and $d\theta_S$ is sufficiently large). The reason why it is a convex symplectic deformation is as follows: We can ensure that θ_S comes from a Stein function ϕ_S on $\mathbb{C}^*$. Also, θ_0 comes from some Stein function $\phi : \mathbb{C}^4 \to \mathbb{R}$, hence $\theta_{M'',0}$ comes from a Stein function $\phi_0 := \phi|_{M''} + P''^*\phi_S$. The 1-form θ'' comes from a Stein function ϕ_1. Hence Θ_t comes from a Stein function of the form

$\phi_t := (1-t)\phi_0 + t\phi_1$. The set of singular points of $\phi_t|_F$ for all t lie inside a compact set K_F (independent of t) for each fibre F. Let K be the union of all the compact sets K_F for each fibre F in M''. We can choose θ_S large enough so that outside some annulus A in $\mathbb{C}^*$, ϕ_t has no singularities in $K \cap P''^{-1}(\mathbb{C}^* \setminus A)$. Also, there are no singularities of ϕ_t outside K. Hence, all the singularities of ϕ_t stay inside some compact set independent of t. This means that ϕ_t is a Stein deformation. This means that $(M'', \theta_{M'',1})$ is convex deformation equivalent to $(M'', \theta_{M'',0})$ which is convex deformation equivalent to (M'', θ'').

Hence on (M'', θ''), we have that the parallel transport maps of P'' are trivial at infinity after a convex symplectic deformation to $(M'', \theta_{M'',1})$.

7.2. **Indices.** Let $P'' : M'' \to \mathbb{C}^*$, $P''(z) = P(z)$. Let F be a smooth fibre of P''. This fibre has a natural exhausting plurisubharmonic function ϕ as in Example 2.8. We can modify ϕ to an exhausting plurisubharmonic function ϕ' which is complete by [25, Lemma 6]. We denote this new Stein manifold by $\widehat{F}$. The following theorem is about indices of a cofinal family of Hamiltonians on $\widehat{F}$.

Theorem 7.2. *There is a cofinal family of Hamiltonians H_λ on $\widehat{F}$ with the following properties:*

1. *There exists some convex symplectic submanifold T of F such that $\widehat{T}$ (the symplectic completion of T) is exact symplectomorphic to $\widehat{F}$.*
2. *$H_\lambda = 0$ on T.*
3. *if y is a periodic orbit of H_λ not in T then $\mathrm{ind}(y) \geq 2$.*
4. *For each $k \in \mathbb{Z}$ there exists an $N > 0$ (independent of λ) such that the number of periodic orbits of H_λ of index k is bounded above by N.*
5. *If we don't count critical points from the interior, then there is exactly one orbit of index 2 and one orbit of index 3 such that the action difference between these two orbits tends to 0 as λ tends to infinity. Also the number of Floer cylinders connecting these orbits is even.*

This theorem is proved by analysing the Conley-Zehnder indices of a Reeb foliation on the Brieskorn sphere $V \cap S$, where S is the unit sphere in $\mathbb{C}^4$. This result needs the following two lemmas:

Lemma 7.3. *$\widehat{F}$ is the completion of some convex symplectic submanifold T with boundary the Brieskorn sphere $V \cap S$.*

Proof. of Lemma 7.3. By Theorem 7.1, we have that $V \setminus 0$ is symplectomorphic to $F \setminus K$ where K is a compact set. Hence there exists a cylindrical end of $\widehat{F}$ which is symplectomorphic to the cylindrical end of V induced by flowing $V \cap S$ by parallel transport. □

Lemma 7.4. *There is a contact form on the Brieskorn sphere $V \cap S$ such that all the Reeb orbits are non-degenerate and they have Conley-Zehnder*

indices ≥ 2. *Also, there is exactly one orbit of index* 2 *and no orbits of index* 3 *and finitely many orbits of degree* k *for each* k.

Proof. In [28] Ustilovsky constructs a contact form such that all the Reeb orbits are non-degenerate and such that their reduced Conley-Zehnder index is $\geq 2(n-2)$ where $n = 3$ in our case. Ustilovsky defines the reduced Conley-Zehnder index to be equal to the Conley-Zehnder index $+(n-3)$. This means that the Reeb orbits have Conley-Zehnder index $\geq n-1 = 2$. He also shows for each $k \in \mathbb{Z}$, there are finitely many orbits of Conley-Zehnder index k. He shows that there are no orbits of odd index and the orbit of lowest index has index 2. □

Proof. of Theorem 7.2.

By Lemma 7.3 $\widehat{F}$ has a convex cylindrical end which is symplectomorphic to $[1,\infty)\times\Sigma$ where Σ is the Brieskorn sphere $V\cap S$. We choose a Hamiltonian which is constant on the interior of F and equal to $h(r)$ on the cylindrical end, where r parameterizes $[1,\infty)$. We also assume that $h'(r)$ is constant and not in the period spectrum of B at infinity. Also, near each orbit in the cylindrical end, we assume that $h'' > 0$. The flow of the Hamiltonian at the level $r = k$ is the same as the flow of $X_H := -h'(k)R$, where R is the Reeb flow. The Conley-Zehnder indices from Lemma 7.4 are computed by trivialising the contact plane bundle. We can trivialise the symplectic bundle by first trivialising the contact plane bundle and then trivialising its orthogonal bundle. We trivialise the orthogonal bundle by giving it a basis $(\frac{\partial}{\partial r}, R)$. The symplectic form restricted to this basis is the standard form:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The Hamiltonian flow in this trivialisation is the matrix:

$$\begin{pmatrix} 1 & 0 \\ h''t & 1 \end{pmatrix}$$

along the orthogonal bundle. This is because R is invariant under this flow and the Lie bracket of $X_H = -h'R$ with $\frac{\partial}{\partial r}$ is $h''R$. The Robbin-Salamon index of this family of matrices is $\frac{1}{2}$. We calculate this index by perturbing this family of matrices by a function $\xi : [0,1] \to \mathbb{R}$ where $\xi(0) = \xi(1) = 0$ as follows:

$$\begin{pmatrix} 1 & \xi(t) \\ h''t & 1 \end{pmatrix}.$$

Choosing ξ so that its derivative is non-zero whenever $\xi = 0$ ensures that the path is generic enough to enable us to compute its Robbin-Salamon index.

Remember that the Robbin-Salamon index of an orbit is equal to the Conley-Zehnder index taken with negative sign. Lemma 7.4 tells us the Conley-Zehnder indices of all the Reeb orbits. The flow $X_H = -h'R$ of the Hamiltonian has orbits in the opposite direction to Reeb orbits. Hence the Robbin-Salamon index (restricted to the contact plane field) of an orbit of

$X_H = -h'R$ is the same as the Conley-Zehnder index of the corresponding Reeb orbit. Hence the Robbin-Salamon index of some orbit of the Hamiltonian on the level set $r = k$ is equal to $C + \frac{1}{2}$ where C is the Conley-Zehnder index of the associated Reeb orbit as calculated in Lemma 7.4. Hence the indices of these orbits are $\geq 2 + \frac{1}{2}$.

The problem is that these orbits are degenerate. This is why their index is not an integer. As in [18, section 3] we can perturb each circle of orbits to a pair of non-degenerate orbits. Let C' be a circle of orbits. We choose a Morse function f on C'. If we flow f along X_H (the Hamiltonian flow of H) we get a time dependent Morse function $f_t = f \circ \phi_{-t}$ (ϕ_t is the Hamiltonian flow). Extend f_t so that it is defined as a function on a neighbourhood of C'. Let $H + f_t$ be our new Hamiltonian. The orbits near C' now correspond to critical points p of f. The Robbin-Salamon index of such an orbit is:

$$i(C') + \frac{1}{2}\text{sign}(\nabla_p^2 f)$$

where $i(C')$ is the Robbin-Salamon index of the manifold of orbits. The symbol 'sign' means the number of positive eigenvalues minus the number of negative eigenvalues. In our case we can choose f so that it has 2 critical points p_1, p_2 such that

$$\text{sign}(\nabla_{p_1}^2 f) = 1,\ \text{sign}(\nabla_{p_2}^2 f) = -1.$$

Hence, if the Conley-Zehnder index of a Reeb orbit C is k, then we can perturb H so that the associated Hamiltonian orbits have Robbin-Salamon index (or equivalently Conley-Zehnder index taken with negative sign) $k+0$ and $k+1$. This means all the non-constant orbits of H have Robbin-Salamon index ≥ 2.

We now need to show that there are a finite number of orbits in each degree. This follows directly from Lemma 7.4 which says that there are finitely many Reeb orbits in each degree. Finally this same lemma says that there is only one Reeb orbit with Conley-Zehnder index 2 and no Reeb orbits with Conley-Zehnder index 3. So the Hamiltonian H has one orbit of Robbin-Salamon index 2 and one orbit of index 3. We can also ensure that the actions of these orbits are arbitrarily close by letting the associated Morse function f be C^2 small. There are an even number of Floer cylinders connecting the orbit of index 3 with the orbit of index 2 by [9, Proposition 2.2]. □

Lemma 7.5. *We have $H^i(M'') = 0$ for $i \geq 2$.*

Proof. $M'' = \mathbb{C}^4 \setminus V$. Theorem 3.1 tells us that V is homeomorphic to $\mathbb{R}^6$. This means that there is a neighbourhood B of V which retracts onto V whose boundary ∂B satisfies $H^i(\partial B) = 0$ for $i \geq 2$. The Mayor-Vietoris sequence involving B, M'' and $B \cup M'' = \mathbb{C}^4$ ensures that $H^i(M'') = 0$ for $i \geq 2$. □

7.3. **Symplectic homology of these varieties.** We wish to show that the symplectic homology of the variety $M'' := \mathbb{C}^4 \setminus V$ has only finitely many idempotents using the results of the previous two sections. We will then show that it has at least two idempotents: 0 and 1. First of all we need the following lemma: We let R be an algebra over $\mathbb{Z}/2\mathbb{Z}$ which is graded by a finitely generated abelian group G. This means that as a vector space, $R = \bigoplus_{g\in G} R_g$ with the property that if $a \in R_{g_1}$ and $b \in R_{g_2}$ then the product ab is contained in $R_{g_1.g_2}$.

Lemma 7.6. *If a is an idempotent in R then $a \in \bigoplus_{g\in G_n} R_g$ where G_n is the subgroup of torsion elements of G.*

Proof. We have $a = a_{g_1} + \cdots + a_{g_n}$ where $g_i \in G$ and $a_{g_i} \in R_{g_i}$. Suppose for a contradiction we have that $a = a^2$ and g_1 is not torsion. Then $a^2 = a_{g_1}^2 + \cdots + a_{g_n}^2$. The group G/G_n is a free $\mathbb{Z}$ algebra, hence there is a group homomorphism $p : G \to G/G_n \to \mathbb{Z}$ such that $p(g_1) \neq 0$. The map p gives R a $\mathbb{Z}$ grading. Let b be an element of R. It can be written uniquely as $b = b_1 + \cdots + b_k$ where b_i are non-zero elements of R with grading $q_i \in \mathbb{Z}$. We can define $f(b)$ as $\min\{|q_j| \neq 0\}$. Note that $f(b)$ is well defined only if at least one of the q_i's are non-zero. Because $p(g_1) \neq 0$, we have that $f(a)$ is well defined and positive. We also have that $f(a^2) \geq 2f(a)$ which means that $a^2 \neq a$. This contradicts the fact that a is an idempotent. □

The vector space $SH_{4+*}(M'')$ is a ring bi-graded by the Robbin-Salamon index and the first homology group. We write $4+*$ here because the unit has Robbin-Salamon index 4. The previous lemma shows us that any idempotent must have grading 4 in $SH_*(M'')$ and be in a torsion homology class.

We have a map $P'' : M'' \to \mathbb{C}^*$. At the end of section 7.1 we had a convex symplectic structure $(M'', \theta_{M'',1})$. Let A be a large annulus in the base $\mathbb{C}^*$ which is a compact convex symplectic manifold. Let $(F'', \theta_{M'',1})$ be a fibre of P''. Choose a compact convex symplectic manifold (with corners) $\bar{M}''$ such that $(\bar{M}'', \bar{P}'' := P''|_{\bar{M}''}, \theta_{M'',1})$ is a compact convex Lefschetz fibration with fibres $\bar{F}''$ and base $A \subset \mathbb{C}^*$. We can also ensure that $\partial\bar{F}''$ is transverse to λ_1 (the associated Liouville vector field of $\mathbb{F}''$) and there are no singularities of λ_1 outside $\bar{F}''$ in F''. Hence the completion of $\bar{M}''$ is $\widehat{M}''$.

Let $(\widehat{E}'', \pi'')$ be the completion of $(\bar{M}'', \bar{P}'', \theta_{M'',1})$ (so that $\widehat{\bar{M}''} = \widehat{E''}$). We wish to use the results of section 5 to show that $SH_*(E'')$ has finitely many idempotents, and hence $SH_*(M'')$ has finitely many idempotents. Let H be a Lefschetz admissible Hamiltonian for $\widehat{E''}$. Let C be the cylindrical end of $\widehat{F''}$. We may assume that this cylindrical end is of the form $(S_V \times [1,\infty), r_F\alpha_F)$ where (S_V, α_F) is the Brieskorn sphere described in 7.2 and r_F is the coordinate for $[1,\infty)$. The Hamiltonian H is of the form $\pi''^* H_{S''} + \pi_1''^* H_{F''}$ as in Definition 2.21. By Lemma 7.6, we have that any idempotent must come from a linear combination of orbits of H in torsion homology classes as long as H is large enough (i.e. it is large enough in some cofinal

sequence of Lefschetz admissible Hamiltonians). Away from $C \times \mathbb{C}^* \subset \widehat{E''}$ we have that the Hamiltonian flow of H is the same as the flow of $L := \pi''^* H_{S''}$. Let X be the Hamiltonian vector field associated to L, and let $X_{S''}$ be the Hamiltonian vector field in $\mathbb{C}^*$ associated to $H_{S''}$. Then the value of X at a point p is some positive multiple of the horizontal lift of $X_{S''}$ to the point p. We can assume that $H_{S''}$ has exactly two contractible periodic orbits of index 0 and 1 corresponding to Morse critical points of $H_{S''}$ (as any Reeb orbit of $\mathbb{C}^*$ is not contractible). We can also make $H_{S''}$ C^2 small away from the cylindrical ends of $\mathbb{C}^*$ so that the only Floer cylinders connecting contractible orbits correspond to Morse flow lines. Hence, any contractible orbit of X must project down to a constant orbit of $X_{S''}$. We let $H_{F''}$ be a Hamiltonian as in Theorem 7.2 above in Section 7.2. We let our almost complex structure J when restricted to $C \times \mathbb{C}^* \subset \widehat{E''}$ be equal to the product almost complex structure $J_F \times J_{\mathbb{C}^*}$ where J_F is an admissible almost complex structure on $\widehat{F}$ and $J_{\mathbb{C}^*}$ is the standard complex structure on $\mathbb{C}^*$. The contractible orbits in this cylindrical end come in pairs (Γ, γ) where Γ is an orbit in $\widehat{F}$ and γ is a contractible orbit in $\mathbb{C}^*$. Because there are only 2 contractible orbits in $\mathbb{C}^*$ and there are finitely many orbits in each degree in $\widehat{F}$, we have finitely many contractible orbits of index 4 for H. Hence:

Theorem 7.7. *The ring $SH_{4+*}(M'')$ has only finitely many idempotents.*

We now wish to show that $SH_*(M'')$ has at least 2 idempotents. To do this we show that $SH_*(M'') \neq 0$, and hence has a unit by [24, Section 8]. This means that $SH_*(M'')$ has 0 and 1 as idempotents. The Hamiltonian H has non-degenerate orbits in $C \times \mathbb{C}^* \subset \widehat{E''}$, so we perturb H away from this set to make all its orbits non-degenerate. In $E'' \subset \widehat{E''}$ we can ensure that H is C^2 small and J is independent of t, hence the only orbits in this region are critical points of H and the only Floer cylinders correspond to Morse flow lines. The orbits corresponding to critical points of H have Robbin-Salamon index ≥ 3 because $H^i(M'') = 0$ for $i > 1$ by Lemma 7.5. Hence all orbits have index ≥ 2. There is only one orbit of index 2. This orbit is in the cylindrical end $C \times \mathbb{C}^* \subset \widehat{E''}$. Hence the orbit is of the form (Γ_m, γ_m) where γ_m has index 0 and Γ_m has index 2. This orbit is closed because there are no orbits of lower index. Suppose for a contradiction this orbit is exact, then there exists a Floer cylinder connecting an orbit β of index 3 with (Γ_m, γ_m). This orbit β must be contractible, so it is either a critical point, or it is of the form (Γ_1, γ_1) in $C \times \mathbb{C}^* \subset \widehat{E''}$. The action of (Γ_m, γ_m) is larger than the action of a critical point and hence β cannot be a critical point. Hence β is of the form (Γ_1, γ_1). Suppose that the index of Γ_1 is 3. We have $\gamma_m = \gamma_1$ and by Theorem 7.2 we can ensure that the action difference between Γ_1 is arbitrarily close to Γ_m. Similarly if γ_1 has index 1 then we can ensure that $\Gamma_m = \Gamma_1$ and the action difference between γ_m and γ_1 is arbitrarily small. This means that the action difference between (Γ_m, γ_m) and (Γ_1, γ_1)

is arbitrarily small. This means that if we have a Floer cylinder connecting (Γ_m, γ_m) and (Γ_1, γ_1) then Gromov compactness ensures that it must stay in the region $C \times \mathbb{C}^* \subset \widehat{E''}$ (Because the action of (Γ_m, γ_m) tends to the action of (Γ_1, γ_1), we get a sequence of Floer cylinders converging to a Floer cylinder of energy 0 which cannot exit $C \times \mathbb{C}^* \subset \widehat{E''}$). Because all the Floer cylinders stay inside $C \times \mathbb{C}^* \subset \widehat{E''}$, the number of Floer cylinders connecting (Γ_m, γ_m) and (Γ_1, γ_1) is equal to the number of Floer cylinders connecting Γ_m and Γ_1 multiplied by the number of Floer cylinders connecting γ_m and γ_1. We need to show that the number of Floer cylinders connecting (Γ_1, γ_1) and (Γ_m, γ_m) is even and by the previous comment, this means we only need to show that the number of Floer cylinders connecting Γ_1 and Γ_m is even or the number of Floer cylinders connecting γ_1 and γ_m is even. But the number of Floer cylinders connecting Γ_1 and Γ_m is even if Γ_1 has index 3 (by part (5) of Theorem 7.2) and similarly γ_1 is closed if it has index 1 (so there are an even number of Floer cylinders connecting γ_1 and γ_m). Hence the number of Floer cylinders connecting these two orbits is even and so (Γ_m, γ_m) is not exact. Hence $SH_*(M'') \neq 0$.

This completes the proof of the main theorem 1.1 subject to checking ring addition under end connect sums.

8. Appendix A: Lefschetz fibrations and the Kaliman modification

Let X,D,M be as in Example 2.8. This means that X is a projective variety with D an effective ample divisor and $M = X \setminus D$ an affine variety. Let Z be an irreducible divisor in X and $q \in (Z \cap M)$ a point in the smooth part of Z. We assume there is a rational function m on X which is holomorphic on M such that $\overline{m^{-1}(0)}$ is reduced and irreducible and $Z = \overline{m^{-1}(0)}$. Let $M' := \text{Kalmod}(M, (Z \cap M), \{q\})$, and let $M'' := M \setminus Z$. Suppose also that $\dim_{\mathbb{C}} X \geq 3$. *Here is the statement of Theorem 2.32: There exist Lefschetz fibrations $E'' \subset E'$ respectively satisfying the conditions of Theorem 1.3 such that E' (resp. E'') is convex deformation equivalent to M' (resp. M'').*

The rest of this section is used to prove this theorem. We will start with several preliminary lemmas.

Lemma 8.1. *There are Stein functions ϕ' (resp. ϕ'') on M' (resp. M'') such that M'' becomes a symplectic submanifold of M'.*

Proof. Let m' be the pullback of m to $\text{Bl}_q X$. Let Z' be the divisor defined by the zero set of m'. Let Z'' be the divisor defined by the zero set of $\frac{1}{m'}$, so that Z' is linearly equivalent to Z''. By abuse of notation, we write D as the total transform of D in $\text{Bl}_q X$.

Let $\tilde{Z}$ be the proper transform of Z. We can choose an effective ample divisor D' with support equal to $\tilde{Z} \cup D$ (as a set) so that $D' - Z''$ is effective. We have that $Y_1 := D'$ and $Y_2 := Y_1 - Z'' + Z'$ are linearly equivalent effective

ample divisors. Let E be a line bundle associated to Y_1 and let s_1, s_2 be sections so that $s_i^{-1}(0) = Y_i$. There is a metric $\|.\|$ of positive curvature on E. We define $\phi' := -dd^c \log(s_1)$ and $\phi'' := -dd^c \log(s_2)$. □

Moving the point q within the smooth part of $Z \cap M$ induces a Stein deformation of M' and M'' by a slight modification of the above lemma.

We now need a technical lemma involving convex symplectic manifolds of finite type. Let (M, θ_1), (M, θ_2) be convex symplectic manifolds. Suppose that $\theta_1 = \theta_2$ inside some codimension 0 submanifold C such that (C, θ_1) is a compact convex symplectic manifold.

Lemma 8.2. *If all the singular points of θ_1 and θ_2 are contained in C, then (M, θ_1) is convex deformation equivalent to (M, θ_2).*

Proof. The interior C^o of C has the structure of a finite type non-complete convex symplectic manifold constructed as follows: The boundary of C has a collar neighbourhood in C of the form $N := (-\epsilon, 1] \times \partial C$, with $\theta_1 = r\alpha$. Here r is the coordinate on $(-\epsilon, 1]$, and α is a contact form on ∂C. We let $\psi : C^o \to \mathbb{R}$ be an exhausting function, which is of the form $h(r)$ on N and such that $h(r) \to \infty$ as $r \to 1$. For some $N \gg 0$, we have that $\psi^{-1}(l)$ is transverse to the associated Liouville field λ_1 for all $l \geq N$. Let ϕ_1 be the function associated to the convex symplectic structure (M, θ_1). We may assume that $\phi_1^{-1}(l)$ is transverse to λ_1 for all $l \geq N$ as well. We can smoothly deform the function ϕ_1 into the function ψ through a series of exhausting functions ϕ_t (the domain of ϕ_t smoothly changes within M as t varies) such that $\phi_t^{-1}(N+k)$ is transverse to λ_1 for each $k \in \mathbb{N}$. This induces a convex symplectic deformation from (M, θ_1) to $(C^o, \theta_1|_{C^o})$. Similarly we have a convex symplectic deformation from (M, θ_2) to $(C^o, \theta_1|_{C^o})$. Hence, (M, θ_1) is convex deformation equivalent to (M, θ_2). □

We need another similar lemma about deformation equivalence.

Lemma 8.3. *Suppose (M, θ_1) and (M, θ_2) are convex symplectic manifolds such that $\theta_1 = \theta_2 + dR$ for some function R, then (M, θ_1) is convex deformation equivalent to (M, θ_2).*

Proof. Let ϕ_1 (resp. ϕ_2) be the function associated with the convex symplectic structure (M, θ_1) (resp. (M, θ_2)). Choose constants $c_1 < c_2 < \cdots$ and $d_1 < d_2 < \cdots$ tending to infinity such that $M_i^1 := \phi_1^{-1}(-\infty, c_i]$ (resp. $M_i^2 := \phi_2^{-1}(-\infty, c_i]$) are compact convex symplectic manifolds. Also we assume that:

$$M_i^1 \subset M_i^2 \subset M_{i+1}^1 \subset M_{i+1}^2$$

for all i. Let $R' : M \to \mathbb{R}$ be a function such that $R' = 0$ on a neighbourhood of ∂M_i^1 and $R' = R$ on a neighbourhood of ∂M_i^2 for all i. Let $\theta_3 := \theta_1 + dR'$. We will show that both (M, θ_1) and (M, θ_2) are convex deformation equivalent to (M, θ_3). Let $R_t : M \to \mathbb{R}$ be a family of functions such that $R_t = 0$ on a neighbourhood of ∂M_i^1 for all i and such that $R_0 = 0$ and $R_1 = R'$. Then $(M, \theta_1 + dR_t)$ is a convex deformation from (M, θ_1) to

(M, θ_3) because $(M_i^1, \theta_1 + dR_t)$ is a compact convex symplectic manifold for all i. Also let $R'_t : M \to \mathbb{R}$ be a family of functions such that $R'_t = R$ on a neighbourhood of ∂M_i^2 and such that $R'_0 = R$ and $R'_1 = R'$. Then $(M, \theta_1 + dR'_t)$ is a convex deformation from (M, θ_2) to (M, θ_3). Hence (M, θ_1) is convex deformation equivalent to (M, θ_2). □

We let E be an ample line bundle on X, and s, t sections of E. We assume that s is non-zero on M. Let t be a holomorphic section of E, and let $p := t/s$ be a map from M to $\mathbb{C}$.

Definition 8.4. *We call* (M, p) *an* **algebraic Lefschetz fibration** *if:*

(1) $\overline{t^{-1}(0)}$ *is smooth, reduced and intersects each stratum of* D *transversally.*
(2) p *has only nondegenerate critical points and there is at most one of these points on each fibre.*

An algebraic Lefschetz fibration (M, p) has a symplectic form ω constructed as in Example 2.8. This means that ω is compatible with p. These are not exact Lefschetz fibrations since the horizontal boundary is not trivial, but they are very useful since our examples arise in this way.

Theorem 8.5. *Parallel transport maps for an algebraic Lefschetz fibration are well defined.*

This is basically proved in [13, section 2], but, there is a subtle distinction between the above theorem and theirs. In [13, section 2], the Stein structure and the Lefschetz fibration are constructed from the same compactification (X, D) of M. In our case they come from different compactifications. The proof can be easily adjusted to this case. This is due to the fact that if we have two different metrics on M induced from compactifications (X_1, D_1) and (X_2, D_2), then the C^2 distance between them is bounded.

We need the following technical lemma so that we can relate algebraic Lefschetz fibrations with ordinary Lefschetz fibrations. We let (E', π'), (E'', π'') be algebraic Lefschetz fibrations such that $\pi''|_{E''} = \pi'$. Let θ' (resp. θ'') be a convex symplectic structure on E' (resp. E'') constructed as in Example 2.8 such that $d\theta'' = d\theta'|_{E''}$. We assume that the real dimension of E' and E'' is 4 or higher.

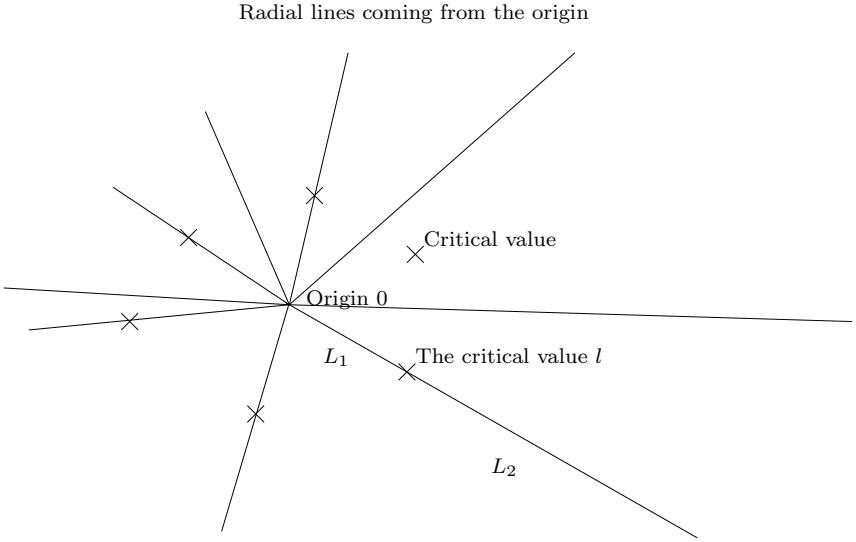
Lemma 8.6. *Suppose that all the singular points of* π' *are contained in* E''. *Then there exists a convex symplectic structure* θ'_1 *(resp.* θ''_1*) on* E' *(resp.* E''*) such that:*

(1) (E', π', θ'_1) *(resp.* (E'', π'', θ''_1)*) are Lefschetz fibrations without boundary.*
(2) $d\theta'_1|_{E''} = d\theta''_1$.
(3) *All the parallel transport maps are trivial on a neighbourhood* N *of* $E' \setminus E''$, *and* $E' \setminus N$ *is relatively compact when restricted to each fibre.*

(4) *For each smooth fibre F' of π', $\theta'|_{F'} = \theta'_1|_{F'} + dR$ for some compactly supported function R. We have a similar statement for (E'', π'').*
(5) *(E', θ'_1) (resp. (E'', θ''_1)) is convex symplectic deformation equivalent to (E', θ') (resp. (E'', θ'')).*

Proof. of Lemma 8.6 We divide this proof into 3 sections. In the first section we construct the Lefschetz fibration without boundary (E', π', θ'_1). In the second section we construct (E'', π'', θ''_1). In section 3 we show that (E', θ'_1) (resp. (E'', θ''_1)) is convex deformation equivalent to (E', θ') (resp. (E'', θ'')).

Step 1 We will use ideas from [23, section 19b]. The map π' has well defined parallel transport maps by Theorem 8.5. We have the same for (E'', π''). Suppose without loss of generality that $0 \in \mathbb{C}$ is a regular point of these fibrations. Let $Q' := \pi'^{-1}(0)$, $Q'' := Q \cap E''$. Consider the family of radial lines in $\mathbb{C}$ coming out of 0. Let L be one of these radial lines which passes through a critical value l of π'. We can write $L = L_1 \cup L_2$ where L_1 is the line joining 0 and l, and $L_1 \cap L_2 = \{l\}$.



We now have vanishing thimbles V_1 and V_2 of l covering L_1 and L_2. A vanishing thimble covering a line L_i is just the set of points in $\pi'^{-1}(L_i)$ which parallel transport along L_i into the critical point associated to the critical value l. Let V be the union of all such thimbles for all radial lines passing through critical values of π'. We can use this to construct a map $\rho : E' \setminus V \to \mathbb{C} \times Q'$. The map is constructed as follows: Let x be a point in $E' \setminus V$. Then we can parallel transport x along a radial line to a point a in Q'. Then $\rho(x) := (\pi'(x), a)$. Let $X = V \cap Q'$ and $W := Q' \setminus X$. Then

$\rho|_{E'\setminus V} : E' \setminus V \to \mathbb{C} \times W$ is a diffeomorphism. Let $\varpi := (\rho|_{E'\setminus V})^{-1}$. Let $\theta_{Q'} := \theta'|_{Q'}$. From now on, if we have a differential form q on $\mathbb{C} \times W$, then we will just write q instead of $\rho^* q$ to clean up notation.

Because parallel transport maps are exact, we have: $\theta'|_{E'\setminus V} = \theta_{Q'} + \kappa' + dR'$ where κ' is a 1-form satisfying $i^*\kappa' = 0$ for all maps i where i is the inclusion map of any fibre of π' into E', and R is some function on $\mathbb{C} \times W$. Let $\bar{f} : W \to \mathbb{R}$ be a function which is equal to 1 near X and is 0 outside some relatively compact neighbourhood of X. We extend $\bar{f}$ by parallel transport along these radial lines to a map $g : E' \setminus V \to \mathbb{R}$. Then we extend g to a map $f : E' \to \mathbb{R}$ as g is constant near V. We will also assume that g is only non-zero inside E'' because parallel transport maps are well defined for (E'', π'', θ''), hence $V \subset E''$. We define

$$\theta'_f := \theta_{Q'} + g\kappa' + d(gR').$$

This form extends over V because $\theta'_f = \theta'$ near V (where $g = 1$). The 1-form θ'_f makes π' into a Lefschetz fibration without boundary where each of the fibres have a convex symplectic structure. We define $\theta'_1 := \theta'_f$.

Step 2 Let $Q'' := \pi''^{-1}(0) \subset E''$. We also have that:

$$\theta'' = \theta_{Q''} + \kappa'' + dR''.$$

Here, κ'' is a 1-form on E'' satisfying $i^*\kappa'' = 0$ for all maps i where i is the inclusion map of a fibre of π'' into E'', and R'' is some function on $E'' \cap (\mathbb{C} \times W)$. Because $d\theta'' = d\theta'$, we have that $d\kappa' = d\kappa''$. This means that $\beta := \kappa' - \kappa''$ is a closed 1-form in E''. We can also show that β is exact as follows: Let $l : \mathbb{S}^1 \to E''$ be a loop. Because we are in dimension 4 or higher, we can perturb the loop so that it doesn't intersect the radial vanishing thimbles described above. We can then deform l using parallel transport to a loop l' contained in a smooth fibre F. We have $\beta|_F = 0$ which means that $\int_l \beta = \int_{l'} \beta = 0$. Hence $\beta = dL$ for some $L : E'' \to \mathbb{R}$. We define:

$$\theta''_f := \theta_{Q''} + g\kappa' + d(gL) + d(gR').$$

We have $d\theta''_f = d\theta'_f$, hence this makes (E'', π'') into a well defined symplectic subfibration of E'. We define $\theta''_1 := \theta''_f$.

Step 3 We can deform $\bar{f}$ through functions which are trivial at infinity to some $\bar{f}'$ where $\bar{f}' = 0$ outside some large compact set, and $(\bar{f}')^{-1}(1)$ contains an arbitrarily large compact set $K \subset F$. We can construct $f' : E' \to \mathbb{R}$ using $\bar{f}'$ in the same way that we constructed f from $\bar{f}$ and the deformation from $\bar{f}$ to $\bar{f}'$ induces a deformation from f to f'. We can choose a convex symplectic structure θ_S on the base so that $(E', \theta'_f + \pi'^*\theta_S)$ and $(E', \theta'_{f'} + \pi'^*\theta_S)$ are convex symplectic manifolds. Hence $\theta'_f + \pi'^*\theta_S$ is convex deformation equivalent to $\theta'_{f'} + \pi'^*\theta_S$. If we choose K large enough we get that $\theta'_{f'} + \pi'^*\theta_S$ is convex deformation equivalent to $(E', \theta' + \pi'^*\theta_S)$ by Lemma 8.2 and Lemma 8.3, and hence is convex deformation equivalent to (E', θ').

Because θ''_f is described in a very similar way to θ'_f, we can use exactly the same argument as above to show that (E'', θ'') is convex deformation equivalent to $(E'', \theta''_f + \pi^*\theta_{S,1})$. The 1-form $\theta_{S,1}$ is a convex symplectic structure on the base making $\theta''_f + \pi^*\theta_{S,1}$ into a convex symplectic structure. □

Let X,D,M be as in Theorem 2.32. This means that Z is an irreducible divisor in X and $q \in (Z \cap M)$ is a point in the smooth part of Z. There is a rational function m on X which is holomorphic on M such that $\overline{m^{-1}(0)}$ is reduced and irreducible and $Z = \overline{m^{-1}(0)}$. We have $M' := \text{Kalmod}(M, (Z \cap M), \{q\})$, and $M'' := M \setminus Z$. We also have $\dim_{\mathbb{C}} X \geq 3$.

Lemma 8.7. *There exist algebraic Lefschetz fibrations*

$$p' : M' \to \mathbb{C},\ p'' : M'' \to \mathbb{C}$$

such that p'' is a subfibration of p' (i.e. $p' \circ (\text{inclusion}) = p''$). Also, if F' (resp. F'') is a page of p' (resp. p''), then F' is the proper transform of F'' in $\text{Bl}_q X$. The singularities of p' are contained in M''.

Proof. Let Q be an effective ample line bundle on X with support equal to D and such that $Q'' := \overline{m^{-1}(0)} + Q$ is ample. Let s'', t'' be sections of Q'' such that ${s''}^{-1}(0) = \overline{m^{-1}(0)} + Q$. We choose t'' such that

$$p'' = \frac{t''}{s''} : M'' \to \mathbb{C}$$

is some algebraic Lefschetz fibration on M''. Let $\bar{F}''$ be the closure of one of the smooth fibres of p'' in M. We can move the point p to somewhere in the smooth part of $\bar{F}'' \cap Z$ as the smooth part of Z is connected (as Z is irreducible); M' is unchanged up to Stein deformation. NB here we use $\dim_{\mathbb{C}} X \geq 2$.

Remember b is the blowdown map $b : \text{Bl}_q X \to X$. Let $s' := b^* s''$ and $t' := b^* t''$. Let Δ be the exceptional divisor $b^{-1}(p)$. The divisor ${s'}^{-1}(0)$ is equal to $\Delta +$ other divisors. We can choose an effective divisor K' with support equal to the boundary divisor D' of M' in $\text{Bl}_q X$ such that $K'' := K' - \Delta$ is ample. Hence, we can choose a meromorphic section h of K'' whose zero set is contained in D', and such that h has a pole of order 1 along the exceptional divisor and such that h is holomorphic away from $D' \cup \Delta$. Let L be the line bundle associated to K''. This means that $s' \otimes h \in H^0(\mathcal{O}(L \otimes b^* Q''))$ is non-zero away from D'. We let $p' := \frac{t' \otimes h}{s' \otimes h}$. This means that $p'|_{M''} = p''$. Because q is in the smooth locus of $\bar{F}'' \cap Z$ and $\bar{F}''$ is transverse to Z, we have that the closure of any smooth fibre of p' intersects each stratum of D' transversally.

We can choose holomorphic coordinates $z_1 \cdots z_n$ on an open set U of p and a holomorphic trivialisation of Q'' such that $s'' = z_1$ and $t'' = z_2$. We then blow up at the point p. Locally around p, we have a subvariety of $U \times \mathbb{P}^n$ defined by $Z_i z_j = Z_j z_i$ where $Z_1 \cdots Z_n$ are projective coordinates for $\mathbb{P}^n$. We choose the chart $Z_1 = 1$. This has local holomorphic coordinates

$z_1, Z_2, Z_3, \cdots, Z_n$. We can choose a trivialisation of K'' so that the section h is equal to $1/z_1$. This means that locally $b^*s'' = Z_1$ and $b^*t'' = Z_2 z_1$. Hence locally, $s' = 1$ and $t' = Z_2$ which means that $p' = Z_2$. This means that p' has no singular points near Δ. Hence p' is also an algebraic Lefschetz fibration which coincides with p'' away from Δ and such all the singular points of p' are the same as the singular points of p''. □

Lemma 8.8. *Let F' (resp. F'') be a fibre of p' (resp. p''). Let K be a compact set in F''. There is a Stein structure J on F' (depending on K) such that any J-holomorphic $u : T \to F'$, where T is a compact Riemann surface with boundary, has the property that $u(T) \subset F''$ if $u(\partial T) \subset F''$.*

Proof. Let G be the closure of F'' in M. Then F' is biholomorphic to $\mathrm{Kalmod}(G, G \cap Z, \{q\})$. Let ϕ_G be a Stein function for G. The compact set K is contained in $\phi_G^{-1}(C)$ for some large C. Let q' be a point outside $\phi_G^{-1}(C)$ which is contained in the smooth part of $G \cap Z$. Because $\dim_{\mathbb{C}} G \cap Z \geq 2$, we can assume that q and q' are in the same irreducible component U of $G \cap Z$. This is where we use the assumption that $\dim_{\mathbb{C}} X \geq 3$. The manifold $G' := \mathrm{Kalmod}(G, G \cap Z, \{q'\})$ is naturally a Stein manifold by Example 2.8. By the comment after Lemma 8.1, we have that G' is Stein deformation equivalent to F' such that it also induces a Stein deformation on M''. The Stein deformation is induced from moving q' smoothly down to q inside the smooth part of $U \subset G \cap Z$ (Note: U is irreducible, hence the smooth part of U is connected). This induces a Stein deformation of M' and M'' which in turn induces a Stein deformation of F' and F''.

From now on we assume that the symplectic structures on F' and G' are complete by [25, Lemma 6]. We can also ensure that the above Stein deformation between F' and G' is complete and finite type by the same lemma, hence by [25, Lemma 5] we have a symplectomorphism $h : F' \to G'$ induced by this Stein deformation. Let $J_{G'}$ be the natural complex structure on G'. Let $J := h^* J_{G'}$. Then if T is a J-holomorphic curve in F' with boundary inside K then $h(T)$ is a holomorphic curve in G' with boundary in $h(K)$. We can blow down this curve to give a holomorphic curve T' in G with boundary in $b(h(K))$. We can ensure that $b(h(K))$ is contained in $\phi_G^{-1}(C)$. If T passes through the blowup of q, then T' passes through q'. This means that $\phi_G \circ T'$ has an interior maximum outside $\phi_G^{-1}(C)$, but this is impossible. Hence the curve T must be contained in F''. □

We can now apply the above lemmas to prove Theorem 2.31. We can apply Lemma 8.6 to p' and p'' to get symplectic fibrations (M', p', θ_1') and (M'', p'', θ_1''). These fibrations are Lefschetz without boundary. We can cut down the fibres to Stein domains $\bar{F}'$ and $\bar{F}''$ where $\bar{F}''$, $\bar{F}'$ are large enough so that the support of all the monodromy maps of (M', p', θ_1') are contained in $\bar{F}''$ and $\bar{F}'' \subset \bar{F}'$. We can also remove the cylindrical end from the base. This will make p' and p'' into Lefschetz fibrations (E', π') and (E'', π'') respectively. Note that if we have a holomorphic curve T in $\bar{F}'$

with boundary in $\bar{F}''$, Lemma 8.8 implies that it is contained in $F'' \cap \bar{F}'$. The Stein maximum principle [18, Lemma 1.5] ensures that T is contained in $\bar{F}''$. Hence we get that Theorem 2.31 is a consequence of Lemma 8.6 and Lemma 8.8.

9. Appendix B: Stein structures and cylindrical ends

The problem with Stein structures is that the complex structure associated with them does not behave well with respect to cylindrical ends. Cylindrical ends here means that near infinity, the convex symplectic manifold is exact symplectomorphic to $(\Delta \times [1, \infty), r\alpha)$ where $r \in [1, \infty)$ and α is a contact form on Δ. The almost complex structure is convex with respect to this cylindrical end if $dr \circ J = -\alpha$. We will deal with this problem in this section.

Let (M, J, ϕ) be a complete finite-type Stein manifold with $\theta = -d^c\phi$ and $\omega = d\theta$. Let $c \gg 0$ be greater than the highest critical value of ϕ.

Theorem 9.1. *There exists a complete finite type convex symplectic structure (M, θ_1) with the following properties:*

1. *It has a cylindrical end with an almost complex structure J_1 which is convex at infinity.*
2. *$J_1 = J$ and $\theta_1 = \theta$ in the region $\{\phi \leq c\}$.*
3. *Any J_1 holomorphic curve with boundary in $\{\phi = c\}$ is contained in $\{\phi \leq c\}$.*
4. *It is convex deformation equivalent to (M, θ) via a convex deformation (M, θ_t) where $\theta_t|_{\{\phi \leq c\}} = \theta|_{\{\phi \leq c\}}$ for $t \in [0, 1]$.*

Proof. Let $\lambda := \nabla\phi$ and $\Delta := \phi^{-1}(c+1)$. We define $G : M \to \mathbb{R}$, $G = \frac{1}{\|\nabla\phi\|^2}$ where $\|.\|$ is the norm defined using the metric $\omega(\cdot, J\cdot)$. Let $\lambda' := G\lambda$. Let $F_t : M \to M$ be the flow of λ'. This exists for all time because ϕ is unbounded and $\mathcal{L}_{\lambda'}\phi = 1$ which implies that ϕ increases linearly with t ($\mathcal{L}$ here means Lie derivative). We have an embedding $\Phi : \Delta \times [1, \infty) \to M$ defined by $\Phi(a, r) = F_{\log r}(a)$ where $a \in \Delta \subset M$ and $r \in [1, \infty)$. Also, $\mathcal{L}_{\lambda'}\theta = G\theta$. Hence, $\Phi^*(\theta) = f\alpha$ where $f : \Delta \times [1, \infty) \to \mathbb{R}$, $f(a, r) := 1 + \int_0^r (G \circ \Phi)(a, t)dt$ and α is the contact form $\theta|_\Delta$ on Δ.

We will now deform the 1-form $f\alpha$ to a 1-form $f'\alpha$ such that $f' = f$ near $r = 1$ and $f' = r$ near infinity. We define θ_1 to be equal to $f'\alpha$ in this cylindrical end and equal to θ away from this end. This means that for r large, we have a cylindrical end with 1-form $f'\alpha = r\alpha$. If we have a function $g : \Delta \times [1, \infty) \to \mathbb{R}$, then $d(g\alpha)$ is non-degenerate if and only if $\frac{\partial g}{\partial r} > 0$. Also, the Liouville vector field associated to $g\alpha$ is $(g/\frac{\partial g}{\partial r})\frac{\partial}{\partial r}$, and hence we have that this Liouville vector field is transverse to every level set $\{r = \text{const}\}$ and pointing outwards. If $(g/\frac{\partial g}{\partial r})$ is bounded above by any polynomial, then the respective Liouville vector field is complete. We define $f' : \Delta \times [1, \infty) \to \mathbb{R}$ such that $f' = f$ near $r = 1$, $f' = r$ near infinity and $\frac{\partial f'}{\partial r} > 0$. This gives a complete finite type convex symplectic structure θ_1 on M as we can extend

$f'\alpha$ outside M as $f' = f$ near $r = 1$. We can join f to f' via a smooth family of functions with f_t ($t \in [0,1]$) where $\frac{\partial f_t}{\partial r} > 0$ and such that $f_t = f$ near $r = 1$. This gives us a convex deformation from θ to θ'.

We now need to construct our almost complex structure J_1. We have $\mathcal{L}_{\lambda'}\phi = Gd\phi(\nabla\phi) = 1$. This means that $\Phi^*(\phi) = \log r$ so the level sets of ϕ coincide with the level sets of $\log r$. By abuse of notation we will just write J for the pullback Φ^*J and we will write ϕ for $\log r$. We have two orthogonal symplectic vector subbundles of the tangent bundle $\Phi^*(TM) = T(\Delta \times [1,\infty))$ whose direct sum is the entire tangent bundle (the symplectic structure we are dealing with here is θ_1). These are: $V_1 := \mathrm{Ker}(\theta_1) \cap \mathrm{Ker}(dr)$ and $V_2 := \mathrm{Span}(\frac{\partial}{\partial r}, X_r)$ where X_r is the Hamiltonian flow of r. The problem is that J is not necessarily compatible with $d\theta_1$, so we need to deform it so that it is. However, near $r = 1$, J is in fact compatible with $d\theta_1$ because $\theta = \theta_1$ in some region $\Xi := \{r \leq 1+\epsilon\}$. Inside Ξ, we have that: $J|_{V_1}$ and $J|_{V_2}$ are holomorphic subbundles of $\Phi^*(TM)$. There exists a complex structure J_{V_1} (resp. J_{V_2}) on the vector bundle V_1 (resp. V_2) compatible with $d\theta_1|_{V_1}$ (resp. $d\theta_1|_{V_2}$) such that, $J_{V_1} = J|_{V_1}$ (resp. $J_{V_2} = J|_{V_2}$) when restricted to Ξ. Because $V_1 \bigoplus V_2 = \Phi^*(TM)$, this gives us an almost complex structure J_1 on $\Phi^*(TM)$ compatible with $d\theta_1$ which is equal to J in the region Ξ. We can choose J_{V_1} and J_{V_2} so that $J_{V_2}(\frac{\partial}{\partial r}) = -\frac{1}{r}X_r$ for $r \gg 0$ and J_{V_1} is invariant under the flow of $\frac{\partial}{\partial r}$ for $r \gg 0$. This ensures that J_1 is convex at infinity. Also, we have that r is plurisubharmonic with respect to J_1 hence any J_1 holomorphic curve with boundary in $\{r = 1\}$ is contained in $\{r \leq 1\}$. Hence property (3) is satisfied. □

10. Appendix C: Transfer maps and handle attaching

The purpose of this section is to show that symplectic homology is additive as a ring under end connect sums. This was already done by Cieliebak in [7] but without taking into account the ring structure. Throughout this section, let (M,θ), (M',θ') be compact convex symplectic manifolds such that M' is an exact submanifold of M of codimension 0. We let $C := N \times [1,\infty)$ be a cylindrical end of M where $\theta = r\alpha$, α is a contact form on N, and r is the coordinate for $[1,\infty)$. Similarly we have a cylindrical end C' of M'. Let $H : M \to \mathbb{R}$ be an admissible Hamiltonian with an almost complex structure J, convex at infinity. Let $SH_*^{(-\infty,a)}(M,H,J)$ be the group generated by orbits of action $< a$. For $b \geq a$, we define

$$SH_*^{[a,b)}(M,H,J) := SH_*^{(-\infty,b)}(M,H,J)/SH_*^{(-\infty,a)}(M,H,J).$$

10.1. **Weak cofinal families.**

Definition 10.1. *We say that the pair (H,J) is* **weakly admissible** *if there exists an $f : N \to \mathbb{R}$ and a constant b such that for $r \gg 0$, $H = re^{-f} + b$ and $d(re^{-f}) \circ J = -\theta$.*

Every admissible pair (H, J) is weakly admissible with $f = \text{const}$. Symplectic homology $SH_*(M)$ is defined as a direct limit of $SH_*(M, H, J)$ with respect to admissible pairs ordered by $\leq$. We wish to replace "admissible" with "weakly admissible". The reason why we wish to do this is because in section 10.3 we carefully construct a cofinal family of weakly admissible pairs to show that symplectic homology behaves well under end connect sums. We construct a partial order $\leq$ on weakly admissible pairs as follows: $(H_0, J_0) \leq (H_1, J_1)$ if and only if $H_0 \leq H_1$. We will show that:

$$SH_*(M) := \varinjlim_{(H,J)} SH_*(M, H, J)$$

where the direct limit is taken over weakly admissible pairs (H, J) ordered by $\leq$. Note that a family of weakly admissible Hamiltonians (H_s, J_s) is cofinal with respect to $\leq$ if the corresponding functions $f_s : N \to \mathbb{R}$ tend uniformly to $-\infty$ as s tends to ∞. In order to ensure that this direct limit exists, we will show that if $(H_0, J_0) \leq (H_1, J_1)$, then there is a natural map of rings $SH_*(M, H_0, J_0) \to SH_*(M, H_1, J_1)$.

This map will be a continuation map. In order for a continuation map to be well defined, we need a family of Hamiltonians T_s joining H_0 and H_1 such that solutions of the parameterized Floer equation $\partial_s u + J_t \partial_t u = \nabla^{g_t} T_s$ joining orbits of H_0 and H_1 stay inside some compact set. To ensure this, we flatten H_i so that it is constant outside some large compact set, and so that all the additional orbits created have very negative action. We do this as follows:

Let D be a constant such that any orbit of H_0 or H_1 has action greater than D. Then $SH_*(M, H_i, J_i) \cong SH_*^{[D,\infty)}(M, H_i, J_i)$. Near infinity, we have that $H_i = R_i + b_i$ where $R_i = re^{-f_i}$. We wish to create a new Hamiltonian K_i such that $K_i = H_i$ on $R_i \leq B$ where $B \gg 0$, and such that K_i is constant in $\{R_i > B + 1\}$ where all the additional orbits have action less than D. We assume that all the orbits of H_0 and H_1 lie in a compact set. We have a cylindrical end $C_i := N_i \times [K, \infty)$ where N_i is the contact manifold $\{re^{-f_i} = 1\}$ with contact form $\theta|_{N_i}$ and R_i is the coordinate for $[K, \infty)$. So, H_i is linear with slope 1 on this cylindrical end. Because all the orbits of H_i lie in a compact set, there are no Reeb orbits of length 1 in the contact manifold N_i. Choose $\epsilon > 0$ such that the length of any Reeb orbit of N_0 or N_1 is of distance more than ϵ from 1. We assume that B is large enough so that H_i is linear with respect to the cylindrical end C_i in $R_i \geq B$ and such that $B\epsilon > -b_i - D$ for $i = 0, 1$. Finally we let K_i be equal to H_i in the region $\{R_i \leq B\}$, and $K_i = k_i(R_i)$ where k_i is constant for $R_i \geq B + 1$, and $k_i = R_i + b_i$ near B and $k_i' \leq 1$. This means that the orbits of K_i in $\{R_i \leq B\}$ are the same as the orbits of H_i, and the orbits of K_i in $\{R_i > B\}$ have action

$$R_i k_i' - k_i < R_i(1 - \epsilon) - R_i - b_i < -B\epsilon - b_i < D.$$

Hence all the orbits of K_i of action greater than D are the same as the orbits of H_i.

We now wish to create an almost complex structure J_i^0 as follows: we let $J_i^0 = J_i$ for $R_i \leq B$ and for $R_i \geq B+1$, we let J_i^0 be convex with respect to the cylindrical end C (i.e. $dr \circ J = -\theta$). Let u be a cylinder or pair of pants satisfying Floer's equation ([24, Formula 8.1]) with respect to (K_i, J_i^0) such that each cylindrical end of u limits to a periodic orbit (or multiple of a periodic orbit in the pair of pants case) inside $\{R_i \leq B\}$. By [3, Lemma 7.2] we have that u is contained in $\{R_i \leq B\}$. Note that we really perturb these Hamiltonians so that all the orbits are non-degenerate, and lie in a compact set. Lemma 7.2 from [3] still works in this case, as we can ensure the Hamiltonian stays the same in the region $B-1 < R_i < B$. From now on if we deal with Hamiltonians which are constant at infinity, we are really perturbing them in such a way that this convexity argument from [3] still holds and such that SH_* is well defined for this Hamiltonian. Hence

$$SH_*(M, H_i, J_i) \cong SH_*^{[D,\infty)}(M, H_i, J_i^0) \cong SH_*^{[D,\infty)}(M, K_i, J_i^0).$$

We wish to create a continuation map:

$$SH_*^{[D,\infty)}(M, K_0, J_0^0) \to SH_*^{[D,\infty)}(M, K_1, J_1^0).$$

There exists a family of Hamiltonians A_s connecting K_0 and K_1, and such that A_s is monotonically increasing and A_s is constant at infinity. We also join J_0^0 and J_1^0 with a family of almost complex structures which are convex with respect to the cylindrical end C (i.e. $dr \circ J = -\theta$). If f_0 and f_1 are constant, then we have a monotone increasing family of admissible Hamiltonians H_s^1 joining H_0 and H_1, and almost complex structures J_s joining J_0 and J_1. The standard continuation map

$$SH_*(M, H_0, J_0) \to SH_*(M, H_1, J_1)$$

involves counting solutions of a parameterized Floer equation with respect to (H_s, J_s). We wish to show that this map is the same as the above continuation map from K_0 to K_1. In order to do this we construct an explicit family (A_S, J_s) of Hamiltonians and almost complex structures so that the continuation map

$$SH_*^{[D,\infty)}(M, K_0, J_0^0) \to SH_*^{[D,\infty)}(M, K_1, J_1^0)$$

coincides with the standard continuation map

$$SH_*(M, H_0, J_0) \to SH_*(M, H_1, J_1)$$

under the isomorphism $SH_*(M, H_i, J_i) \cong SH_*^{[D,\infty)}(M, K_i, J_i^0)$. The Hamiltonians H_s^1 are of the form $h_s(r)$ for $r \geq P$ where h_s' is constant. We assume that the almost complex structures J_s are convex with respect to the cylindrical end C for $r \geq P$. Hence [3, Lemma 7.2] ensures all the Floer trajectories with respect to (H_s, J_s) stay inside the compact set $r \leq P$. Also there is a constant P' such that K_i is a function of r for $r \geq P' \geq P$. The definition of K_i depends on a parameter B which can be arbitrarily large.

We can choose B large enough so that $K_i = H_i$ in the region $r \leq P'$. We choose the functions A_s joining K_0 and K_1 so that A_s is a function of r for $r \geq P'$. We can also assume that $J_s^0 = J_s$. In order to show that the maps are the same, we need to show that any Floer trajectory associated to (A_s, J_s^0) connecting orbits inside $\{r \leq P\}$ is contained in $\{r \leq P\}$. This follows from [3, Lemma 7.2].

10.2. **Transfer maps.** In this section we will construct a natural ring homomorphism: $SH_*(M) \to SH_*(M')$. We say that H is called **transfer admissible** if $H \leq 0$ on M'. We have: $SH_*(M) = \varinjlim_{(H,J)} SH_*(M, H, J)$ where the direct limit is taken over transfer admissible Hamiltonians ordered by $\leq$.

Lemma 10.2. *We have an isomorphism of rings,*

$$\varinjlim_{(H,J)} SH_*^{[0,\infty)}(M, H, J) \cong SH_*(M')$$

where the direct limit is taken over transfer admissible Hamiltonians.

Proof. We construct a particular cofinal family of transfer admissible Hamiltonians H_i and show the above isomorphism of rings. We can embed $\widehat{M'}$ into $\widehat{M}$ by Lemma 2.5. Our cylindrical end C' is then a subset of M. We assume that the action spectrum $\mathcal{S} := \mathcal{S}(\partial M')$ is discrete and injective. Let $k : \mathbb{N} \to \mathbb{R} \setminus \mathcal{S}$ be a function such that $k(i)$ tends to infinity as i tends to infinity. Let $\mu : \mathbb{N} \to \mathbb{R}$ be defined by $\operatorname{dist}(k(i), \mathcal{S})$ ($\operatorname{dist}(a, B)$ is the shortest distance between a and B). From now on we just write k instead of $k(i)$, and similarly for μ.

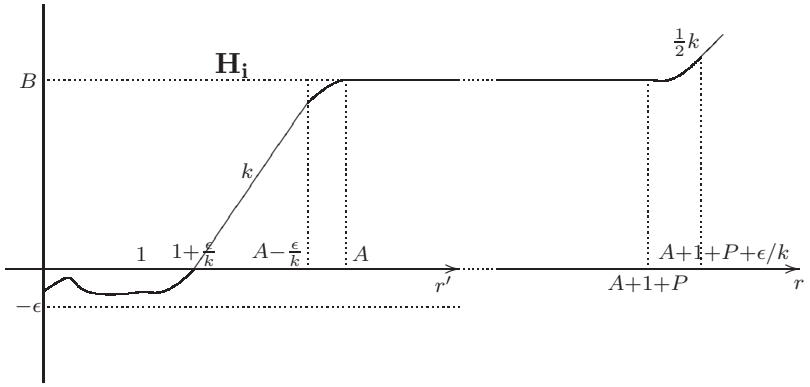
Define:

$$A = A(i) := 6k/\mu > k > 1.$$

We can assume that $A > k > 1$ because we can choose $k(i)$ to make $\mu(i)$ arbitrarily small whilst $k(i)$ is large. We also let $\epsilon := \epsilon(i)$ tend to 0 as i tends to infinity. We assume that $H_i|_{M'} \leq 0$, and has slope $k(i)$ on $1+\epsilon/k \leq r' \leq A - \epsilon/k$. We also assume that on $1 \leq r' \leq A$, $H_i = h(r')$ for some function h where h has non-negative derivative $\leq k$. For $A \leq r' \leq A+1$, we assume that H_i is constant. Let B be this constant. B is arbitrarily close to $k(A-1)$. We can assume that $B \notin \mathcal{S}$. We now describe H_i on the cylindrical end C. We keep H_i constant until we reach $r = A+1+P$ where P is some constant large enough so that $\{r' \leq 1\} \subset \{r \leq P\}$. This means that $\{r' \leq A+1\} \subset \{r \leq A+1+P\}$ as long as we embed C' in the same way as Lemma 2.5. We then let H_i be of the form $f(r)$ for $r \geq A+1+P$ where $f' < \frac{1}{2}k$ and has slope $\frac{1}{2}k$ for $r > A+1+P+\epsilon/k$.

Here is a picture of what we have:

Figure 10.3.



The action of an orbit on a level set $r' = a$ is $h'(a)a - h(a)$. The orbits near $r' = 1$ have positive action less than or equal to k. Let p be a point on an orbit o lying in the region $A - \epsilon/k \leq r' \leq A$. The slope $h'(r')$ of H_i at p is $\leq k - \mu$. Hence, the orbits near $r' = A$ have action $\leq (k-\mu)A - B = -\mu A + k \to -\infty$ as $i \to \infty$. So, we can assume that these orbits have negative action. Also all the orbits in $\{r' \geq A, r \leq A + 1 + P\}$ are fixed points, so have action $-B < 0$. Finally, the orbits in $r > A + 1 + P$ have action: $\leq \frac{1}{2}k.(A + 1 + P) - B = -\frac{1}{2}kA + \frac{3}{2}k + \frac{1}{2}Pk \to -\infty$ as $i \to \infty$. Hence, we can assume that all the orbits of H_i of non-negative action lie in $r' < 1 + \epsilon/k$.

We now need to show that any differential connecting two orbits of non-negative action is contained entirely in $r' < 1 + \epsilon/k$. By [3, Lemma 7.2], any differential connecting orbits of non-negative action must be contained in $r' < 1 + \epsilon/k$ as all the orbits of non-negative action are contained in this region. A similar application of this lemma ensures that a pair of pants satisfying Floer type equations with similar Hamiltonians where the ends converge to orbits of non-negative action must be contained in $r' < 1 + \epsilon/k$. This means that we have maps $SH_*^{[0,\infty)}(M, H_i, J) \cong SH_*(M', H_i', J')$, where $H_i' : \widehat{M'} \to \mathbb{R}$ has slope k. Taking direct limits gives us a ring isomorphism

$$\varinjlim_{(H,J)} SH_*^{[0,\infty)}(M, H, J) \cong SH_*(M').$$

□

This lemma enables us to define a transfer map:

$$SH_*(M) \cong \varinjlim_{(H,J)} SH_*(M, H, J) \to \varinjlim_{(H,J)} SH_*^{[0,\infty)}(M, H, J) \cong SH_*(M').$$

A Hamiltonian is called **weakly transfer admissible** if it is weakly admissible and is negative when restricted to M'. We can combine the above results with the results of section 10.1 to construct the above transfer map using a cofinal family of weakly transfer admissible Hamiltonians. We will need to construct a cofinal family of weakly transfer admissible Hamiltonians

in section 10.3 to show that a particular transfer map is an isomorphism of rings.

Here is an application of the transfer map:

Lemma 10.4. *If $SH_*(M) = 0$, then $SH_*(M') = 0$.*

Proof. We have a commutative diagram:

$$\begin{array}{ccc} H^{n-*}(M) & \xrightarrow{a} & H^{n-*}(M') \\ \downarrow b & & \downarrow d \\ SH_*(M) & \xrightarrow{c} & SH_*(M') \end{array}$$

Suppose for a contradiction $SH_*(M') \neq 0$. Then [24, Section 8] says that the map d is non-zero in degree n. Also the map a is an isomorphism in degree n. Hence $d \circ a$ is non-zero, and so $c \circ b = d \circ a$ is non-zero. This means that $SH_*(M) \neq 0$ and we get a contradiction. □

Corollary 10.5. *If M is subcritical and $SH_*(M') \neq 0$, then M' cannot be embedded in M as an exact codimension 0 submanifold. In particular, if $H_1(M') = 0$ then M' cannot be symplectically embedded into M.*

Proof. By one of the applications of [19], we have that $SH_*(M) = 0$ because M is subcritical. The result follows from the above lemma. □

10.3. **Handle attaching.** In this section we will prove Theorem 2.20. Here is the statement of Theorem 2.20: Let M, M' be finite type Stein manifolds of real dimension greater than 2, then
$SH_*(M \#_e M') \cong SH_*(M) \times SH_*(M')$ as rings. Also the transfer map $SH_*(M \#_e M') \to SH_*(M)$ is just the natural projection

$$SH_*(M) \times SH_*(M') \twoheadrightarrow SH_*(M).$$

To prove this, we will show that attaching a symplectic 1-handle to a compact convex symplectic manifold of dimension ≥ 2 does not change symplectic homology. We will describe in more detail what it means to attach a symplectic 1-handle later. In fact we show that if A is a compact convex symplectic manifold, and A' is equal to A with a symplectic 1-handle attached, then the natural transfer map $SH_*(A') \to SH_*(A)$ is an isomorphism. This proves Theorem 2.20 for the following reason: The Stein manifold M (resp. M') is convex deformation equivalent to $\widehat{N}$ (resp. $\widehat{N'}$) where N (resp. N') is a compact convex symplectic manifold. The end connect sum $M \#_e M'$ is convex deformation equivalent to $\widehat{N''}$ where N'' is the disjoint union $N \coprod N'$ with a symplectic 1-handle joining each connected component. The symplectic homology of $N \coprod N'$ is the direct product $SH_*(N) \times SH_*(N')$. Also, the transfer map $SH_*(N \coprod N') \to SH_*(N)$ is the natural projection $SH_*(N) \times SH_*(N') \twoheadrightarrow SH_*(N)$. Hence assuming that adding a 1-handle

does not change symplectic homology, we have that $SH_*(N'')$ is isomorphic to $SH_*(N) \times SH_*(N')$, hence $SH_*(M\#_e M')$ is isomorphic to this product. This implies that $SH_*(M\#_e M') \cong SH_*(M) \times SH_*(M')$ because M (resp. M') is convex deformation equivalent to N (resp. N'). Also the natural transfer map from $SH_*(M\#_e M')$ to $SH_*(M)$ is the natural projection $SH_*(M) \times SH_*(M') \twoheadrightarrow SH_*(M)$ because this corresponds to the composition of maps

$$SH_*(N'') \xrightarrow{\cong} SH_*(N \coprod N') \xrightarrow{\cong} SH_*(N) \times SH_*(N') \twoheadrightarrow SH_*(N)$$

We will now describe handle attaching in detail as in [7, Section 2.2]. The paper [10] or [8, Theorem 9.4] ensures that this construction corresponds to attaching a Stein 1-handle. We will define ϕ, p_i, q_i, X, ω, $\psi(x,y)$ as in [7, Section 2.2]. We will now remind the reader what these variables are: We set $k = 1$, so we are describing 1-handles only. We let $\mathbb{R}^{2n}$ have coordinates $(p_1, q_1, \ldots, p_n, q_n)$.

$$\omega := \sum_i dp_i \wedge dq_i,$$

$$\phi := \frac{1}{4}\sum_{i=1}^{n-1}\left(q_i^2 + p_i^2\right) + q_n^2 - \frac{1}{2}p_n^2,$$

$$X := \nabla\phi = \frac{1}{2}\sum_{i=1}^{n-1}\left(\frac{\partial}{\partial q_i} + \frac{\partial}{\partial p_i}\right) + 2\frac{\partial}{\partial q_n} - \frac{\partial}{\partial p_n}.$$

ψ is a function of x and y where:

$$x := \sum_{i=1}^{n-1}\left(A_i q_i^2 + B_i p_i^2\right),$$

$$y := B_n p_n^2,$$

and $A_i, B_i > 0$ are constants. It satisfies $X.\psi > 0$ provided that:

$$\frac{\partial\psi}{\partial x} \geq 0,\ \frac{\partial\psi}{\partial y} \leq 0,\ \frac{\partial\psi}{\partial x}(x,0) > 0,\ \frac{\partial\psi}{\partial y}(y,0) < 0,$$

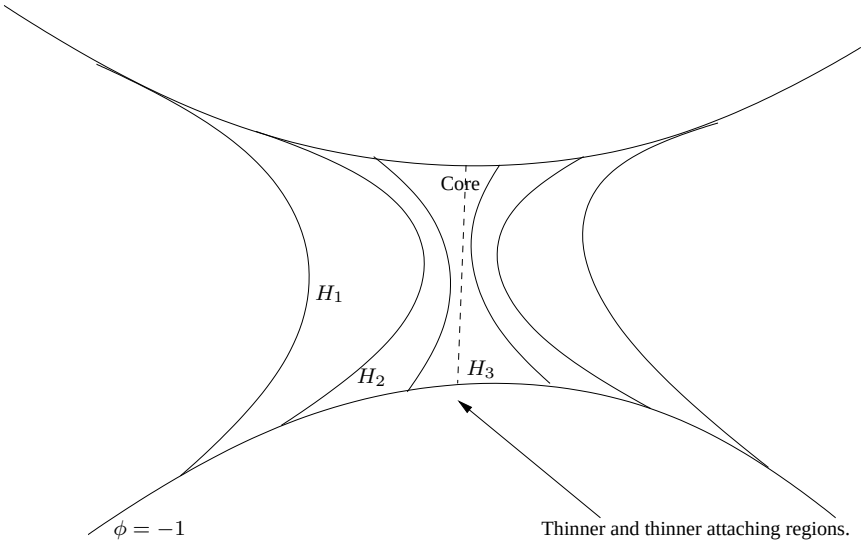
and the partial derivatives are not simultaneously 0. We can choose ψ so that the level sets $\{\phi = -1\}$ and $\{\psi = 1\}$ agree outside some compact set. This ensures that when we glue the handle onto our convex symplectic manifold, it still has a smooth boundary so we don't have to smooth the handle once we have attached it.

The handle $H = H_1^{2n} := \{\phi \geq -1\} \cap \{\psi \leq 1\}$. We define $\partial^- H$ to be the boundary $\{\phi = -1\}$. We can ensure that the only 1-periodic orbit of ψ is the critical point at the origin by [7, Section 2.2]. We wish to construct a family of 1-handles (constructed in the same way as H) $(H_l)_{l\in\mathbb{N}}$ with the following properties:

(1) $H_{l+1} \subset H_l$
(2) The attaching region $\partial^- H_{l+1}$ is a subset of $\partial^- H_l$

(3) As l tends to infinity, H_l converges uniformly to the core of the handle.

This can be done by shrinking ψ.



Let M be a compact convex symplectic manifold. After a deformation, we can assume that the boundary of M has a region A which is contactomorphic to the attaching region $\partial^- H_1$. We can also ensure that the period spectrum of ∂M is discrete and injective (we might have to deform the region A and ϕ slightly and hence all the handles). We can use the region A to attach the handle H_l to M to create a new compact convex symplectic manifold $M_l := M \cup_{\partial^- H_l} H_l$. We have that $M_{l+1} \subset M_l$ and the boundary of each M_l is transverse to the Liouville vector field on M_1. Let K be an admissible Hamiltonian on $\widehat{M}$. We assume that K has slope S in a neighbourhood of ∂M. We choose l large enough so that the attaching region $P := \partial^- H_l$ has the property that a Reeb flowline outside P intersecting P twice has length greater than S. We can now extend the Hamiltonian K to a Hamiltonian $K' : M_l \to \mathbb{R}$ using the function $B\psi$ where B is some constant. Hence ∂M_l is a level set of K' and K' is linearly increasing on a neighbourhood of ∂M_l. Hence we can extend K' to an admissible Hamiltonian on $\widehat{M_l}$. The periodic orbits of K' are the same as the periodic orbits of K with an extra fixed point at the origin of the 1-handle. We can ensure that the index of the extra fixed point at the origin of the 1-handle has index strictly increasing as S increases (see the last part of the proof of Theorem 1.11 in [7, Section 3.4]). Because ∂M_l is transverse to the Liouville field of ∂M_1, we have that $\widehat{M_1} = \widehat{M_l}$ and K' is weakly admissible. Hence we have a cofinal family of weakly transfer admissible Hamiltonians K'. The only orbit outside $M \subset \widehat{M_1}$ has arbitrarily large index, hence these K''s induce a transfer isomorphism of rings $SH_*(M) \to SH_*(M_1)$. This proves Theorem 2.20.

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